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Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 3

(vector fields, tangent bundle)

due 09.11.2011

Exercise 1 **5 points**

On $\mathbb{S}^2 = \{x = (x_0, x_1, x_2) \mid \|x\| = 1\} \subset \mathbb{R}^3$ we consider coordinates given by the stereographic projection from the north pole $N = (1, 0, 0)$:

$$y_1 = \frac{x_1}{1 - x_0}, \qquad y_2 = \frac{x_2}{1 - x_0}.$$

Let the vector fields X and Y on $\mathbb{S}^2 \setminus \{N\}$ be defined in these coordinates by

$$X = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}, \qquad Y = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the sterographic projection from the south pole $S = (-1, 0, 0)$.

Exercise 2 **5 points**

Let M be a smooth manifold of dimension n. For each chart $\varphi\colon U \to \mathbb{R}^n$ we define $TU := \bigcup_{p \in U} T_pM \subset TM$ and a map $\Phi\colon TU \to \mathbb{R}^{2n}$ by

$$T_pM \ni X_p \mapsto (\varphi(p), X_p\varphi) \in \mathbb{R}^{2n}.$$

Show:

i) Φ is injective and $\Phi(TU) = \varphi(U) \times \mathbb{R}^n$.

ii) If $\Psi\colon TV \to \mathbb{R}^{2n}$ is another such map corresponding to chart $\psi\colon V \to \mathbb{R}^n$ and $W := U \bigcap V \neq \emptyset$, then

$$\Psi \circ \Phi^{-1}\colon \Phi(TW) \to \Psi(TW)$$

is a diffeomorphism.

Remark: Let M be a smooth manifold of dimension n and $\{(U_j, \varphi_j)\}_{j\in J}$ an atlas of M. For each chart (U_j, φ_j) there is a corresponding mapping $\Phi_j \colon TU_j \to \mathbb{R}^{2n}$ defined as in Exercise 2. If we introduce a topology on TM by demanding all those maps to be homeomorphisms, then TM becomes a topological manifold. In exercise 2 we have shown that these maps are actually C^∞–compatible, hence they define a smooth structure on TM.
From now on, we will consider TM as the smooth $2n$–manifold with this structure.

Exercise 3 **5 points**

Prove that the tangent bundle of a product of manifolds is diffeomorphic to the product of the tangent bundles of the manifolds. Deduce that the tangent bundle of a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$.