Technische Universität Berlin
Institut für Mathematik
Sullivan / Knöppel
http://page.math.tu-berlin.de/~sullivan/L/11W/DG2/ WS 11

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 14

(Lie algebras, subgroups, and homomorphisms)

due 15.2.2012

Exercise 1 **5 points**

For any skew–commutative algebra $(\mathfrak{g}, [\,,\,])$, we define the map $ad\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ by $ad\,(x)\,(y) = [x, y]$. Verify that the validity of the identity

$$[ad\,(x)\,, ad\,(y)] = ad\,([x, y])$$

(where, as usual, the bracket on $\operatorname{End}(\mathfrak{g})$ is the commutator) is equivalent to the validity of the Jacobi identity

$$[[x, y]\,, z] + [[y, z]\,, x] + [[z, x]\,, y] = 0$$

for all $x, y, z \in \mathfrak{g}$.

Exercise 2 **5 points**

Let G be a Lie group and let H be an algebraic subgroup. Show that if there is an open neighborhood U of e in G so that $H \cap U$ is a smooth embedded submanifold of G, then H is a Lie subgroup of G.

Exercise 3 **5 points**

Let G be a Lie group. A vector field Y on G is right invariant if Y is R_a–related to itself for each $a \in G$. Prove that the set of right invariant vector fields on G forms a Lie algebra and is naturally isomorphic as a vector space to T_eG. Let $\varphi\colon G \to G$ be the diffeomorphism defined by $\varphi\,(a) := a^{-1}$. Prove that if X is left invariant vector field on G, then $\varphi_* X$ is the right invariant vector field whose value at e is $-X_e$. Prove that $X \mapsto \varphi_* X$ gives a Lie algebra isomorphism of the Lie algebra of left invariant vector field on G to the Lie algebra of right invariant vector fields on G.