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http://page.math.tu-berlin.de/~sullivan/L/11W/DG2/ WS 11

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 10

(connections, parallel transport)

due 18.1.2012

Exercise 1 **5 points**

Show that if $E_1, \ldots, E_n$ is a parallel frame field along a differentiable curve γ in a smooth manifold M and $X(t) = X_{\gamma(t)}$ is a vector field along the curve defined by $X(t) = \sum_{i=1}^{n} a_i(t) E_{i\gamma(t)}$, then

$$\frac{DX}{dt} = \sum_{i=1}^{n} \frac{da_i}{dt} E_{i\gamma}.$$

Exercise 2 **5 points**

Let (M, g) and (N, h) be Riemannian manifolds and ∇ and $\tilde{\nabla}$ its Riemannian connections, respectively. Let $f\colon M \to N$ be an isometry, i.e. $f^*h = g$. Show:

$$\tilde{\nabla}_{f_*X} f_*Y = f_* \nabla_X Y.$$

Exercise 3 **5 points**

Let γ be a curve in a smooth manifold M and $\gamma(t_0)$ a point on it. The mapping $P_{\gamma(t),\gamma(t_0)}\colon T_{\gamma(t_0)}M \to T_{\gamma(t)}M$ defined by $P_{\gamma(t),\gamma(t_0)} X_{\gamma(t_0)} = X_{\gamma(t)}$, where $X_{\gamma(t)}$ is the unique extension of $X_{\gamma(t_0)}$ to a parallel vector field along γ, is called the parallel transport from $X_{\gamma(t_0)}$ to $X_{\gamma(t)}$. Show:

a) The parallel transport is a linear isomorphism, and if $X(t) = X_{\gamma(t)}$ is a vector field along γ, then

$$\left.\frac{DX}{dt}\right|_{t=t_0} = \lim_{t \to t_0} \frac{P_{\gamma(t_0),\gamma(t)} X(t) - X(t_0)}{t - t_0}.$$

b) If M is an oriented Riemannian manifold with Riemannian connection ∇, then $P_{\gamma(t),\gamma(t_0)}$ is an isometry preserving orientation.