Exercise 1:
Let $U \subset \mathbb{R}^2$ be an open subset in the plane. Prove that a function $f : U \to \mathbb{R}$ is locally constant (i.e. each point has a neighborhood on which $f$ is constant) if and only if $f$ is constant on each connected component.

Exercise 2:
On which of the following open subsets $U \subset \mathbb{R}^2$ in the plane is $\omega_\theta$ exact (i.e. there is a smooth angle function $\theta$ with $\omega_\theta = d\theta$ on $U$)? Motivate your answer.

(a) The union of the upper half plane and the right half plane.

(b) The complement of the negative $y$-axis.

(c) The complement of the line segment joining $P$ to 0, where $0 \neq P \in \mathbb{R}^2$.

(d) The complement of the line through $P$ and 0, where again $0 \neq P \in \mathbb{R}^2$.

(e) An annulus centered at 0, that is, the set $\{(x, y) \in \mathbb{R}^2 \mid n < x^2 + y^2 < m\}$, for some $0 < n < m$.

Exercise 3:
Let $P \in \mathbb{R}^2$, then $\omega_P$ will be the closed 1-form corresponding to “$d\theta$”around $P$. If $P = (x_0, y_0)$ then we’ll define it as follow on $\mathbb{R}^2 \setminus \{P\}$:

$$\omega_P := -\frac{(y - y_0)dx + (x - x_0)dy}{(x - x_0)^2 + (y - y_0)^2}$$

Given two distinct points $P, Q \in \mathbb{R}^2$ prove that $\omega_P - \omega_Q$ is not exact on $\mathbb{R}^2 \setminus \{P, Q\}$. Show, however, that it is exact on $\mathbb{R}^2 \setminus \overline{PQ}$, where $\overline{PQ}$ denotes the line segment from $P$ to $Q$.

Exercise 4:
Suppose $U \subset \mathbb{R}^2 \setminus \{0\}$ is a subset in which there exists a continuous angle function $\theta$ (for which $d\theta = \omega_\theta$). Suppose $\gamma$ is a closed path in $U$. Show that the winding number $W(\gamma, 0)$ equals 0.