Exercise Sheet 1
Due in tutorials on 27 October 2010

Exercise 1:
On any set \( X \) we can define the cofinite topology as follows. \( C \subset X \) is closed if and only if \( C = X \) or \( C \) is finite. Prove that this topology is well defined. Is \( X \) with this topology Hausdorff?

Exercise 2:
Let \( f : X \rightarrow Y \) be a continuous map between topological spaces. Prove that if \( X \) is compact, then \( f(X) \) is compact.

Exercise 3:
Give \( \{0, 1\} \) the discrete topology. On the product space \( X := \mathbb{R} \times \{0, 1\} \) we define the following relation:
\[
(x, a) \sim (y, b) \iff (x, a) = (y, b) \text{ or } x = y < 0
\]
Prove that \( X/\sim \) is the union of two open sets homeomorphic to \( \mathbb{R} \). Is it Hausdorff? (Motivate your answer.)

Exercise 4:
Let \( f : X \rightarrow Y \) be a map between topological spaces.
We say that \( f \) is open if for every open set \( U \subset X \), \( f(U) \) is open in \( Y \).
Similarly, \( f \) is closed if for every closed set \( V \subset X \), \( f(V) \) is closed in \( Y \).
For a continuous bijection \( f : X \rightarrow Y \), prove that the following conditions are equivalent:

1. \( f \) is a homeomorphism
2. \( f \) is open
3. \( f \) is closed