TECHNISCHE UNIVERSITÄT BERLIN WS 10
Institut für Mathematik Abgabe: ?
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http://www.math.tu-berlin.de/~sullivan/L/10W/DG2

7. Übung Differentialgeometrie II: Mannigfaltigkeiten

(preparation for the test)

Hausaufgaben

1. Aufgabe (0 Punkte)

Let real projective space $\mathbb{RP}^n$ be the quotient set of $\mathbb{R}^{n+1}\backslash 0$ by the equivalence relation:

$$(y_1, \dots, y_{n+1}) \sim (\lambda y_1, \dots, \lambda y_{n+1})$$

, for any non-zero real number λ. Let $[y_1 : \dots : y_{n+1}]$ denote the equivalence class of $(y_1, \dots, y_{n+1})$. Consider the maps

$$\phi_i : V_i := \{[y_1 : \dots : y_i : \dots : y_{n+1}] | y_i \neq 0\} \to \mathbb{R}^n$$
$$[y_1 : \dots : y_i : \dots : y_{n+1}] \mapsto \frac{1}{y_i}(y_1, \dots, \hat{y}_i, \dots, y_{n+1})$$

for all $i = 1, \dots, n+1$. Show that the collection $(\phi_i, V_i)_{1 \leq i \leq n+1}$ defines a smooth structure on $\mathbb{RP}^n$. Is $\mathbb{RP}^n$ compact? What happenes if one replaces the real numbers by the complex numbers? What is the dimension of $\mathbb{CP}^n$?

2. Aufgabe (0 Punkte)

Show that $\mathbb{RP}^1$ and S^1 are diffeomorphic.

3. Aufgabe (0 Punkte)

Let A be real symmetric $n \times n$ matrix and $b \neq 0$. Show that

$$M := \{x \in \mathbb{R}^n | x^T A x = b\}$$

is a $n-1$ dimensional submanifold of $\mathbb{R}^n$. If all eigenvalues of A are greater zero then M is diffeomorphic to $\mathbb{S}^n$.

4. Aufgabe (0 Punkte)

Let $f : M \to N$ be an embedding with $f(p) = q$. Show

1. $f^* : C^\infty(q) \to C^\infty(p)$ is onto.

2. $f_* : T_pM \to T_qN$ is injective.

5. Aufgabe (0 Punkte)

On $\mathbb{R}^3$ consider the vector fields

$$X := x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \quad Y := x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \quad Z := [X, Y].$$

Compute the coordinate expression of Z and describe (in words, geometrically) the flows of all three vector fields.

6. Aufgabe (0 Punkte)

Let X be a vector field on $\mathbb{S}^2$, which is never tangent to the equator $\mathbb{S}^1 := \mathbb{S}^2 \cap (\mathbb{R}^2 \times \{0\}) \subset \mathbb{R}^3$. Show that the integral curves of X intersect the equator at most once.

7. Aufgabe (0 Punkte)

We consider functions P, Q defined on $\mathbb{R}^4$ and coordinates x, y, u, v. Let

$$\omega_1 := dx - Pdu + Qdv, \quad \omega_2 := dy - Qdu - Pdv.$$

Show that $\omega_1 = \omega_2 = 0$ defines a two-plane distribution. Determine the conditions on P and Q under which this distribiution is completely integrable.

8. Aufgabe (0 Punkte)

Let $i : \mathbb{S}^3 \to \mathbb{R}^4$ denote the inclusion map.

1. Show that for every $p \in \mathbb{S}^3$ the kernel of the map $i^* \colon T_p^*\mathbb{R}^4 \to T_p^*\mathbb{S}^3$ equals $\lambda(x^1dx^1 + x^2dx^2 + x^3dx^3 + x^4dx^4), \lambda \in \mathbb{R}$.

2. Prove that the restriction $i^*\sigma$ of

$$\sigma = x^1\, dx^2 - x^2\, dx^1 + x^3\, dx^4 - x^4\, dx^3$$

   to $\mathbb{S}^3$ is nowhere zero.

Gesamtpunktzahl: 0