3. Übung Differentialgeometrie II: Mannigfaltigkeiten
(vector fields, tangent bundle)

Hausaufgaben

1. Aufgabe (5 Punkte)
On $\mathbb{S}^2 = \{(x_0, x_1, x_2): x_0^2 + x_1^2 + x_2^2 = 1\}$ we consider coordinates given by the stereographic projection from the north pole

$$y_1 = \frac{x_0}{1 - x_2}, \quad y_2 = \frac{x_1}{1 - x_2}.$$

Let $X, Y$ be the vector fields defined on $\mathbb{S}^2 \setminus \{0, 0, 1\}$ which are given in these coordinates by

$$X = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}, \quad Y = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the stereographic projection from the south pole.

2. Aufgabe (5 Punkte)
Let $M^n$ be a smooth manifold. For a chart

$$\phi = (\phi^1, \ldots, \phi^n): U \rightarrow \mathbb{R}^n$$

we define

$$\Phi: TU = \bigcup_{p \in U} T_p M \rightarrow \mathbb{R}^{2n},$$

$$X_p \mapsto (\phi^1(p), \ldots, \phi^n(p), X_p(\phi^1), \ldots, X_p(\phi^n)),$$

where $X_p \in T_p M$

1. Show that $\Phi$ is injective and $\Phi[TU] = \phi[U] \times \mathbb{R}^n$. 

2. Given another chart $\psi: V \to \mathbb{R}^n$ and $\Psi: TV \to \mathbb{R}^{2n}$ as above, show that

$$\Psi \circ \Phi^{-1}: \Phi[T(U \cap V)] \to \Psi[T(U \cap V)]$$

is a diffeomorphism.

After defining a Hausdorff topology on $TM$ such that for every chart $\phi: U \to \mathbb{R}^n$ of $M$ the corresponding map $\Phi$ is a homeomorphism of an open subset of $TM$ onto its image (which we will now assume has been done), the functions $\Phi$ define a smooth structure on $TM$.

From now on, we will consider $TM$ as a smooth $2m$-manifold with this structure.

3. **Aufgabe**

(5 Punkte)

Prove that the tangent bundle of a product of manifolds is diffeomorphic to the product of the tangent bundles of the manifolds. Deduce that the tangent bundle of a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$.

Gesamtpunktzahl: 15