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# 10. Übung Differentialgeometrie II: Mannigfaltigkeiten

(Cartan's magic formula, exterior derivative )

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## Hausaufgaben

## 1. Aufgabe (5 Punkte)

Let $M$ be a smooth manifold equippded with a riemannian metric $g$ and a volume form $\omega$ and $f : M \to M$ be a diffeomorphism. Show that the following conditions are equivalent:

1. $f$ is an isometry

2. $f$ is conformal (angle preserving) i.e. $f^*g = \lambda g$ for some positive smooth function $\lambda$ and volume preserving, i.e $f^*\omega = \omega$.

## 2. Aufgabe (5 Punkte)

We define the Lie derivative of differential forms similary to the Lie derivative of vector fields: if $\omega$ is a form and $X$ is a vector field on a manifold $M$, the Lie derivative of $\omega$ along $X$ at a point $p$ is given by

$$(L_X(\omega))_p := \lim_{t\to 0} \frac{(\psi^t)^*\omega_p - \omega_p}{t},$$

where $\psi^t$ is the flow defined by $X$ in a neighborhood of $p$. Show the following properties:

1. $L_X(f) = X(f)$, provided $f$ is a 0-form (smooth function) on M:

2. $L_X(\omega_1 + \omega_2) = L_X(\omega_1) + L_X(\omega_2)$ and $L_X(f\omega) = X(f)\omega + fL_X(\omega)$;

3. $L_X(d\omega) = d(L_X(\omega))$:

4. Cartan's magic formula: $L_X = i_X \circ d + d \circ i_X$ where $i_X$ is the contraction operator defined by

$$(i_X\omega)_p(V_1(p), ..., V_{k-1}(p)) := \omega_p(X(p), V_1(p), ..., V_{k-1}(p)).$$

## 3. Aufgabe

(5 Punkte)

Show that for any $k$-form $\omega$ on $M$, and smooth vector fields $V_1, ..., V_{k+1}$ on $M$, then $d\omega$ is given by

$$\begin{aligned} d\omega(V_1, ..., V_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}V_i(\omega(V_1, ..., \hat{V}_i, .., V_{k+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([V_i, V_j], V_1, ..\hat{V}_i, .., \hat{V}_i, ...V_{k+1}]), \end{aligned}$$

where ˆ means that the corresponding entry is omitted, and $[.,.]$ denotes the Lie bracket.

Gesamtpunktzahl: 15