Exercise 1: Area of convex planar polygons. (6 pts)

Let \( p = (p_1, ..., p_n = p_0) \) be a convex planar polygon. Let \( \phi_i \) be the turning angle at \( p_i \) and let \( \ell_i = |p_{i+1} - p_i| \) be length of the \( i \)-th edge.

In lecture, we discussed an analog of the Heintze-Karcher inequality for \( p \): in terms of the curvature density \( \kappa_i \) along the \( i \)-th edge defined as \( \frac{\tan(\phi_i/2) + \tan(\phi_{i+1}/2)}{\ell_i} \), we bounded the area enclosed by \( p \) as follows:

\[
\text{Area}(p) \leq \sum_{i=0}^{n-1} \ell_i / \kappa_i.
\]

Show that we have equality here if and only if \( p \) has an inscribed circle (tangent to each of the \( n \) edges).

(Hint: the medial axis (or cut locus) of \( p \) is always a tree with \( n \) “leaf edges” along the angle bisectors at the vertices \( p_i \). Our estimate is sharp if and only if this tree has no internal edges, i.e. just one internal vertex.)

Exercise 2: Gauss curvature of one quarter of the unit ball. (6 pts)

Consider the set \( \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0\} \subset \mathbb{R}^3 \), which is one quarter of the unit ball. Find the Gauss curvature of its piecewise smooth boundary.

Exercise 3: Gauss curvature of a polyhedron. (6 pts)

Let \( v \) be a vertex of a regular icosahedron. Its five neighbors form a regular pentagon in a plane \( P \). Now reflect \( v \) (and the five incident triangles) across \( P \) to get a new (nonconvex) polyhedron (an “icosahedron with a dimple”).

1. What is the total Gauss curvature?
2. What is the total absolute Gauss curvature?

Due: Tutorial on 24.06.10