Exercise Sheet 3

Exercise 1: Curves of finite total curvature in $\mathbb{R}^3$. (4 pts)

Find an explicit example of a curve of finite total curvature in $\mathbb{R}^3$ whose projection to the $xy$-plane has infinite total curvature. In particular, find a sequence of points $p_k = (x_k, y_k, z_k)$ approaching $p_0 = (0, 0, 0)$ such that the “infinite polygon” $p_1, p_2, \ldots, p_0$ has finite total curvature but its projection does not.

Exercise 2: Cauchy-Crofton formula. (4 pts)

1. For $k < d$ prove the following analog of the Cauchy-Crofton formula:

   There is some constant $c_k^d$ such that given any curve $\gamma$ in $\mathbb{R}^d$, its length is $c_k^d$ times the average length of its projections to $k$-planes.

2. Find $c_1^3$ and $c_2^3$.

(Note: for $j < k < d$ we have $c_j^d = c_j^k c_k^d$, by projecting a curve in $\mathbb{R}^d$ first to a $k$-plane and then to a $j$-plane.)

Exercise 3: Closed convex curve with constant width. (2 pts)

Let $\gamma$ be a closed convex curve of constant width $d$. Prove that the length of $\gamma$ is $\pi d$, just as for a circle.

Due: Tutorial on 11.05.10