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Geometry I
WS 09/10

http://www.math.tu-berlin.de/~sullivan/L/09W/Geo1/

Exercise Sheet 5

Exercise 1: Circles. (4 pts)

Let C denote the Euclidean circle with center $(0, s)$ and radius s, where $0 < s < 1/2$.

1. Compute the center and radius of the hyperbolic circle represented by C in the Poincaré disk model $\mathbb{D}^2$.
2. Let C' denote the hyperbolic circle in $\mathbb{D}^2$ with center $(0, 0)$ and the same hyperbolic radius as C. Then as Euclidean circles, which has larger radius, C or C'? Why?

Exercise 2: Möbius transformations. (4 pts)

Consider the half-plane model $\mathbb{H}^2_+ := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$. Let $C \subset \mathbb{H}^2_+$ be a hyperbolic line (i.e. either a semicircle orthogonal to the real axis or a vertical straight line). Let γ be a Möbius transformation defined by

$$\gamma(z) := \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{R},\ ad - bc > 0.$$

1. Prove that $\gamma(C)$ is a hyperbolic line.
2. Let $C' \neq C$ be a hyperbolic line. Prove that there exists a Möbius transformation that maps C to C'.

Exercise 3: Projective lines in $P^2(\mathbb{R})$. (4 pts)

1. Find the equation of the projective line in $P^2(\mathbb{R})$ joining $A = [1, 2, 0]$ to $B = [1, 0, 1]$.
2. Let $\ell \subset P^2(\mathbb{R})$ be the projective line through $[0, 1, 1]$ and $[1, 0, 1]$; let $\ell' \subset P^2(\mathbb{R})$ be the projective line through $[1, 1, 1]$ and $[0, 2, -1]$. Find the homogeneous coordinates of the point $\ell \cap \ell'$.

Exercise 4: Projective transformations in $P^2(\mathbb{R})$. (4 pts)

1. Find a projective transformation $\tau_1 : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ which maps the points

   $$P_0 = [1, 2, 0], \quad P_1 = [0, 1, 0], \quad P_2 = [-1, 0, 2], \quad P_3 = [-1, 3, 4],$$

   to the points

   $$Q_0 = [1, 0, 0], \quad Q_1 = [1, 1, 0], \quad Q_2 = [0, -1, 1], \quad Q_3 = [1, 1, 1],$$

   respectively.

Turn over

2. Find a projective transformation $\tau_2 : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ which maps the points P_0, P_1, P_2, P_3 from 1. to the points P_3, P_2, P_1, P_0 (i.e. τ_2 interchanges these points).

Exercise 5 (optional): Riemannian metrics. (extra 4 pts)

Consider the following hyperbolic models:

$$\mathbb{J} := \{(x_1, \dots, x_{n+1}) \,|\, x_1^2 + \dots + x_{n+1}^2 = 1,\ x_{n+1} > 0\},$$
$$\mathbb{L} := \{(x_1, \dots, x_{n+1}) \,|\, x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}.$$

$\mathbb{J}$ is the Hemisphere model and $\mathbb{L}$ is the Hyperboloid model. The corresponding Riemannian metrics are:

$$ds_{\mathbb{J}}^2 = \frac{dx_1^2 + \dots + dx_{n+1}^2}{x_{n+1}^2}, \qquad ds_{\mathbb{L}}^2 = dx_1^2 + \dots + dx_n^2 - dx_{n+1}^2.$$

The isometry between $\mathbb{L}$ and $\mathbb{J}$ is the central projection from the point $(0, 0, \dots, 0, -1)$:

$$\alpha : \mathbb{L} \to \mathbb{J}, \quad (x_1, \dots, x_{n+1}) \mapsto \left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}, \frac{1}{x_{n+1}} \right).$$

Prove that $\alpha^*(ds_{\mathbb{J}}^2) = ds_{\mathbb{L}}^2$, where $\alpha^*(ds_{\mathbb{J}}^2)$ is the pullback of $ds_{\mathbb{J}}^2$ by the isometry α.

Due: Tutorial on 27.11.09