TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Dr. John M. Sullivan
Dott. Matteo Petrera

Geometry I
WS 09/10

http://www.math.tu-berlin.de/~sullivan/L/09W/Geo1/

Exercise Sheet 11

Exercise 1: Alternative definition of the cross-ratio for a conic. (4 pts)

Given a non-degenerate conic $\gamma \subset \mathbb{R}P^2$, the cross-ratio of four points $P_i \in \gamma$, $i = 1, 2, 3, 4$, is defined by $\mathrm{cr}(P_1, P_2, P_3, P_4) := \mathrm{cr}(QP_1, QP_2, QP_3, QP_4)$, where Q is an arbitrary point on γ.

Suppose that ℓ is an arbitrary tangent line to $\gamma \subset \mathbb{R}P^2$. Let P_i, $i = 1, 2, 3, 4$, be four points on γ. Let ℓ_i be the tangent lines to γ at P_i and let $R_i := \ell_i \cap \ell$. Prove that

$$\mathrm{cr}(P_1, P_2, P_3, P_4) = \mathrm{cr}(R_1, R_2, R_3, R_4).$$

Exercise 2: Inversion of circles. (4 pts)

Prove the following claims:

1. Let γ be a circle with centre O. The inverse of any line ℓ, not through O, is a circle through O and the diameter through O is perpendicular to ℓ.
2. A pair of intersecting circles α and β, with common points O and P inverts into a pair of intersecting lines α' and β' through the inverse point P'.

Exercise 3: Inversion of circles. (4 pts)

Let γ be a circle with centre O and let δ be a circle, not through O, having centre C. Let C' be the inverse of C with respect to γ. Let the circle δ' be the inverse of δ with respect to γ. Show that C' is *not* the centre of δ'.
(Hint: show that any circle β through C' and O is orthogonal to δ' by observing that the image β' of β under inversion in γ is a line orthogonal to δ).

Exercise 4: The Möbius group. (4 pts)

The Möbius group $\mathrm{M\ddot{o}b}(N)$ is defined as the group of maps $\mathbb{R}^N \cup \{\infty\} \to \mathbb{R}^N \cup \{\infty\}$ generated by reflections in hyperplanes $\{x \in \mathbb{R}^N : \langle v, x - a\rangle = 0, v, a \in \mathbb{R}^N\}$,

$$x \mapsto x - \frac{2\langle v, x - a\rangle}{\langle v, v\rangle} v,$$

and by inversions in hyperspheres $\{x \in \mathbb{R}^N : \|x - c\| = r^2, c \in \mathbb{R}^N, r > 0\}$,

$$x \mapsto c + \frac{r^2}{\|x - c\|^2}(x - c).$$

Turn over

Show that the following maps are in the Möbius group by writing them as a composition of reflections in hyperplanes and inversions on hyperspheres:

1. The translations in $\mathbb{R}^N$, $x \mapsto x + b$, $b \in \mathbb{R}^N$;
2. The scaling transformations in $\mathbb{R}^N$, $x \mapsto \lambda x$, $\lambda \in \mathbb{R} \setminus \{0\}$;
3. The rotations in $\mathbb{R}^3$.

Due: Tutorial on 05.02.10