TECHNISCHE UNIVERSITÄT BERLIN
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Geometry I
WS 09/10

http://www.math.tu-berlin.de/~sullivan/L/09W/Geo1/

Exercise Sheet 10

Exercise 1: Cross-ratio for a conic. (4 pts)

Given a non-degenerate conic $\gamma \subset \mathbb{R}P^2$, the cross-ratio of four points $P_i \in \gamma$, $i = 1, 2, 3, 4$, is defined by $\mathrm{cr}(P_1, P_2, P_3, P_4) := \mathrm{cr}(QP_1, QP_2, QP_3, QP_4)$, where Q is an arbitrary point on γ.

Let $\gamma \subset \mathbb{R}P^2$ be a non-degenerate conic. Let $P, Q, R \in \mathbb{R}P^2$ be such that γ is tangent to PQ at $Q \in \gamma$ and to PR at $R \in \gamma$. Prove that for any $A, B \in \gamma$ the following formula holds:

$$[\mathrm{cr}(Q, R, A, B)]^2 = \mathrm{cr}(PQ, PR, PA, PB).$$

Exercise 2: Triangle circumscribed around a conic. (4 pts)

Let $\triangle ABC \subset \mathbb{R}P^2$ be a triangle circumscribed around a non-degenerate conic $\gamma \subset \mathbb{R}P^2$. The lines CB, AC, AB meet γ at P_1, P_2, P_3 respectively. Show that AP_1, BP_2, CP_3 are concurrent.

Exercise 3: Canonical forms of quadrics. (6 pts)

Classify and find the canonical form of the following quadrics in $\mathbb{R}^3$ (up to Euclidean motion) :

1. $\sigma_1 : \; 6xz + 8yz - 5x = 0;$
2. $\sigma_2 : \; 6xz + 8yz - 5 = 0;$
3. $\sigma_3 : \; 3x^2 + 2y^2 + 2xz + 3z^2 - 4 = 0.$

Exercise 4: Circular cone. (2 pts)

Decide which of the following equations describes the circular cone that is obtained when one rotates the line $\ell := \{[x, y, z] : x = 0, z = 2y\}$ around the z -axis:

$$x^2 + 4y^2 = z^2, \quad 4(x^2 + y^2) - z^2 = 0, \quad 2(x^2 + y^2) - z^2 = 0, \quad z = 4(x^2 + y^2).$$

Exercise 5: Intersections. (4 pts)

The quadrics $\sigma_1 : \; z = x^2 + y^2$ and $\sigma_2 : \; z = x^2 - y^2$ are both examples of paraboloids. Find the equations of planes $\pi_1, \pi_2, \pi_3, \pi_4$ (each parallel to some coordinate plane) such that:

1. $\sigma_1 \cap \pi_1$ is a parabola;
2. $\sigma_1 \cap \pi_2$ is a circle;
3. $\sigma_2 \cap \pi_3$ is a hyperbola;
4. $\sigma_2 \cap \pi_4$ is a pair of lines.

Due: Tutorial on 29.01.10