Differentialgeometrie II

Übungsblatt 9

Due on Wednesday 28.01.2009

1 Aufgabe

Let $\phi \in \bigwedge^r(\mathbb{R}^n)$. Show that ϕ is closed iff $\int_{\partial B} \phi = 0$ for any $r+1$ dimensional submanifold B of $\mathbb{R}^n$.

2 Aufgabe

Let M be an n-dimensional oriented manifold, with orientation determined by $\Omega \in \bigwedge^n(M)$.

- Show that if M is Remannian, the metric determines uniquely a vector field X defined at each point $p \in \partial M$ and such that X_p is orthogonal to $T_p\partial M$, inward pointing (cfr. remark VI.4.5) and of unit length.
- Show that in the general case where M is not Remannian, it is possible to find a vector field X defined at each point $p \in \partial M$ such that it is inward pointing (use partition of unity).
- Show that $(-1)^{n-1}\iota_X(\Omega) \in \bigwedge^{n-1}(\partial M)$ determines on ∂M the same orientation as in theorem VI.4.4.

3 Aufgabe

Let S^n be the unit sphere, viewed as a submanifold of $\mathbb{R}^{n+1}$. We endow $\mathbb{R}^{n+1}$ with standard coordinates $x^1, \dots, x^{n+1}$ and standard volume form $\Omega = dx^1 \wedge \dots \wedge dx^{n+1}$.

Consider B_{n+1} (the unit ball) and define on $S^n = \partial B_{n+1}$ an orthnormal inward pointing vector field X as in the first part of problem 2.

Write the expression for $\omega := (-1)^n \iota_X(\Omega) \in \bigwedge^n(S^n)$.

Consider the map $\sigma S^n \to S^n$ sending $(x^1, \dots, x^n, x^{n+1})$ to $(-x^1, \dots, -x^n, -x^{n+1})$.

Show that $\sigma^*(\omega) = (-1)^{n+1}\omega$.

4 Aufgabe

Let S^n be as in the preceding problem.

Consider the equivalence relation $\sim$ on S^n defined as $x \sim y$ iff $x = \sigma(y)$, for $x, y \in S^n$.

Let $N := S^n/\sim$ be the quotient manifold and $\pi : S^n \to N$ the quotient map. Show that

- $\forall f \in \bigwedge^0(N),\ \sigma^* \circ \pi^*(f) = \pi^*(f) \in \bigwedge^0(S^n)$;
- $\forall f \in \bigwedge^0(N),\ \sigma^* \circ \pi^*(df) = \pi^*(df) \in \bigwedge^1(S^n)$;
- $\forall \phi \in \bigwedge^r(N),\ \sigma^* \circ \pi^*(\phi) = \pi^*(\phi). \in \bigwedge^r(S^n)$.

Conclude that for n even, N is not orientable.

Hint: remember from the preceding problem that $\sigma^*(\omega) = (-1)^{n+1}\omega$.