Differentialgeometrie II

Übungsblatt 7

1 Aufgabe 1

Let $End(TM)$ denote the space of C^∞ fields of linear transformations of the tangent bundle of the manifold M. In other words, $\phi \in End(TM)$ assigns to each point $p \in M$ a linear map $\phi_p : T_pM \to T_pM$ such that ϕ_p varies smoothly.

This last sentence means the following: Let (U, ψ) be a chart with associated local frame $\{E_i\}$ and co-frame $\{\omega_i\}$. Denote by $T^{i,j}$ the field of linear transformations associating to each $p \in U$ the operator $T_p^{i,j} : T_pM \to T_pM$ defined by $T_p^{i,j}(E_{k|_p}) = \delta_{j,k} E_{i|_p}$. We say that ϕ is smooth when for any chart $\phi = \sum_{i,j} \alpha_{i,j} T^{i,j}$ (on U) and the $\alpha_{i,j} = \omega_i(\phi(E_j))$ are functions belonging to $C^\infty(U)$.

Prove that $End(TM)$ and $\mathcal{T}_1^1(M)$ are isomorphic.

Remark You can follow the hint given in Boothby exercise V.5.3 to find an isomorphism between $End(T_pM)$ and $\mathcal{T}_1^1(T_pM)$. There $\langle \ , \ \rangle \in \mathcal{T}_1^1(V)$ denotes the 1-covariant 1-contravariant tensor taking value $\bar{w}(v)$ for generic $v \in V$ and $\bar{w} \in V^*$.

Don't forget to show that the isomorphism sends C^∞ fields into C^∞ fields.

2 Aufgabe 2

Define $End(TM)$ as the linear space of $C^\infty(M)$-linear maps from $\mathcal{X}(M)$ to $\mathcal{X}(M)$. Show that this definition is equivalent to the one given above.

3 Aufgabe 3

Let $\mathcal{S}$ and $\mathcal{A}$ be, respectively, the symmetrizing and alternating mappings defined in Boothby, definition V.5.6.

Show that their compositions are zero: $\mathcal{SA} = \mathcal{AS} = 0$.

4 Aufgabe 4 (Boothby, problem V.6.3)

Let $\phi_i \in \bigwedge^1(V) = V^*$ and $v_j \in V$, with $i, j \in \{1, \ldots, r\}$.
Show that $\phi_1 \wedge \cdots \wedge \phi_r(v_1, \ldots, v_r) = det(\phi_i(v_j))$.