Differentialgeometrie II
Übungsblatt 6

Due 17 December 2008

1 Aufgabe (8 pts)

Consider $\mathbb{R}^2$ as a Remannian manifold with usual coordinates x, y, associated frame $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and metric given by the 2-form $\Phi(\ ,\ ) = dy^2 + (a^2 + x^2)dx^2$, where a is a positive number.

Consider the curvilinear triangle with vertices $A = (0,0), B = (1, \frac{a^2}{2}), C = (1, -\frac{a^2}{2})$, and described by the following curves

- α given by $y = \frac{a^2}{2}x^2$;
- β given by$y = -\frac{a^2}{2}x^2$;
- γ given by $x = 1$.

Calculate the perimiter of the triangle. Calculate the angles at the vertices.

2 Aufgabe (5 pts)

Consider the parametrized surface (in $\mathbb{R}^3$) $h(u,v) = \gamma(u) + vp$, where $p = (p_1, p_2, p_3)$ is a fixed point in $\mathbb{R}^3$ and $\gamma(u) = (y_1(u), y_2(u), y_3(u))$ is a C^∞ curve in $\mathbb{R}^3$. Analogously to problem 1 in last week's list, determine the components of the Remannian metric induced on the surface by the standard metric on $\mathbb{R}^3$. Write the expression of the metric form for the surface in terms of the 1-forms du, dv.

3 Aufgabe (7pts)

Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$ with standard coordinates and standard metric form $dx^2 + dy^2$. Show that $\omega_1 = x\,dx + y\,dy$ and $\omega_2 = y\,dx - xdy$ are two linearly independent 1-forms. Consider the diffeomorphism $\phi : M \to M$ defined by $\phi(x,y) = (\frac{x}{(x^2+y^2)}, \frac{y}{(x^2+y^2)})$. Write the explicit expression for $\phi^*(dx^2 + dy^2)$ (i.e. as a linear combination of $dx\,dy, dx^2, dy^2$.) Does ϕ leave the metric invariant? *Suggestion* It is quicker to use polar coordinates to do the calculation, bu you can use x, y variables as well, with some patience. Explain why also $\phi^*(\omega_1)$ and $\phi^*(\omega_2)$ are linearly independent.

Denote by X_{ω_1} and X_{ω_2} the vector fields dual to ω_1 and ω_2 with respect to the standard metric. In other words, $\forall p \in M, \forall Y_p \in T_pM,\ \omega_{ip}(Y_p) = \langle X_{\omega_i p}, Y_p \rangle$, where $\langle , \rangle$ denotes the standard metric $dx^2 + dy^2$ evaluated on the two vectors $X_{\omega_i p}, Y_p$.

Is it true that $X_{\phi^*(\omega_i)} = \phi_*(X_{\omega_i})$?