Differentialgeometrie II
Übungsblatt 6

Due 17 December 2008

1 Aufgabe (8 pts)

Consider $\mathbb{R}^2$ as a Riemannian manifold with usual coordinates $x, y$, associated frame $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and metric given by the 2-form $\Phi(\cdot, \cdot) = dy^2 + (a^2 + x^2)dx^2$, where $a$ is a positive number.

Consider the curvilinear triangle with vertices $A = (0, 0), B = (1, \frac{a^2}{2}), C = (1, -\frac{a^2}{2})$, and described by the following curves

- $\alpha$ given by $y = \frac{a^2}{2}x^2$;
- $\beta$ given by $y = -\frac{a^2}{2}x^2$;
- $\gamma$ given by $x = 1$.

Calculate the perimeter of the triangle. Calculate the angles at the vertices.

2 Aufgabe (5 pts)

Consider the parametrized surface (in $\mathbb{R}^3$) $h(u, v) = \gamma(u) + vp$, where $p = (p_1, p_2, p_3)$ is a fixed point in $\mathbb{R}^3$ and $\gamma(u) = (y_1(u), y_2(u), y_3(u))$ is a $C^\infty$ curve in $\mathbb{R}^3$. Analogously to problem 1 in last week’s list, determine the components of the Riemannian metric induced on the surface by the standard metric on $\mathbb{R}^3$. Write the expression of the metric form for the surface in terms of the 1-forms $du, dv$.

3 Aufgabe (7 pts)

Let $M = \mathbb{R}^2 \setminus \{(0, 0)\}$ with standard coordinates and standard metric form $dx^2 + dy^2$. Show that $\omega_1 = x\, dx + y\, dy$ and $\omega_2 = y\, dx - x\, dy$ are two linearly independent 1-forms. Consider the diffeomorphism $\phi : M \to M$ defined by $\phi(x, y) = (\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}})$. Write the explicit expression for $\phi^*(dx^2 + dy^2)$ (i.e. as a linear combination of $dx\, dy, dx^2, dy^2$.) Does $\phi$ leave the metric invariant? Suggestion It is quicker to use polar coordinates to do the calculation, but you can use $x, y$ variables as well, with some patience. Explain why also $\phi^*(\omega_1)$ and $\phi^*(\omega_2)$ are linearly independent.

Denote by $X_{\omega_1}$ and $X_{\omega_2}$ the vector fields dual to $\omega_1$ and $\omega_2$ with respect to the standard metric. In other words, $\forall p \in M, \forall Y_p \in T_p M, \omega_i p (Y_p) = \langle X_{\omega_i} p, Y_p \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard metric $dx^2 + dy^2$ evaluated on the two vectors $X_{\omega_1} p, Y_p$.

Is it true that $X_{\phi^*(\omega_1)} = \phi_*(X_{\omega_1})$?