Differentialgeometrie II
Übungsblatt 5

Due 3 December 2008

1 Aufgabe

Consider the following C^∞ vector fields defined on $M = \mathbb{R}^3$:

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}; \quad Y = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}; \quad Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

- Compute the three products $[X, Y],\ [X, Z],\ [Y, Z]$;
- Describe the flows of the vector fields $X,\ Y,\ Z,\ [X, Y],\ [X, Z],\ [Y, Z]$;
- The result for $[X, Z]$ and $[Y, Z]$ could have been predicted without explicitely calculating the Lie bracket, but by observing instead the relative behaviour of the flows X, Y, Z. Explain how.

2 Aufgabe

Let $X, X', Y \in \mathcal{X}(M)$ be C^∞ vector fields on a manifold M, and fix a point $p \in M$. Show that $X_p = X'_p$ does not (in general) imply $(L_X Y)_p = (L_{X'} Y)_p$.

3 Aufgabe

Let $X, Y \in \mathcal{X}(M)$ be C^∞ vector fields. Let $f, g \in C^\infty(M)$. Show that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

4 Aufgabe

Prove that the following two definitions of involutive distribution are equivalent:

(a) For all $X, Y \in \mathcal{X}(M)$ which belong to Δ it follows that $[X, Y]$ belongs to Δ;

(b) For any point $p \in M$, there exists an open neighbourhood $U_p \ni p$ and a local frame $\{X_i \in \mathcal{X}(U_p)\}$ for Δ such that $[X_i, X_j] = \sum_k c^k_{i,j} X_k$, with $c^k_{i,j} \in C^\infty(U_p)$.

Remark Let M be an m-dimensional manifold. By an n-dimensional distribution Δ we mean the following (cf. Boothby IV.8.2):

- for each point $p \in M$ We have an n-dimensional subspace $\Delta_p \subset T_p(M)$;
- for each point $p \in M$ there is an open $U_p \subset M$ and a local frame (for Δ) $\{X_i \in \mathcal{X}(U_p)\}$ defined on U_p, i.e. n distinct C^∞ vector fields defined on U_P such that for any $q \in U_p$, $\{X_{i\ q}\}$ is a basis for Δ_q.
- A field X is said to belong to Δ if $\forall p \in M$, $X_p \in \Delta_p$. Thus, the restriction of X on U_p satisfies $X|_{U_p} = \sum_i \alpha_i X_i$, where α_i belong to $C^\infty(U_p)$.

Hint for (b) $\Rightarrow$ (a): For any U_p write the restrictions of two generic X, Y which belong to Δ as a $C^\infty(U_p)$-linear combination of the X_i. Use the result of the preceding exercise.