Differentialgeometrie II
Übungsblatt 2

Due 5 November 2008

Remark: The notes available on the course webpage might be useful. In particular, the second set of notes contains hints for problem 2.

1 Aufgabe

(a) Show that S^1, viewed as the set of points $\{x, y \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is regularly embedded in $\mathbb{R}^2$.

(b) Consider $M_2(\mathbb{R})$ (the set of 2×2 real matrices) as a differentiable manifold with atlas given by a single chart: the natural identification with $\mathbb{R}^4$. Show that $\mathcal{SO}(2, \mathbb{R})$ (the subset of orthogonal 2×2 matrices with determinant 1) is homeomorphic to S^1.

(c) Show that $\mathcal{SO}(2, \mathbb{R}) \subset M_2(\mathbb{R})$ has the 1-submanifold property (in in the sense of Boothby, def III.5.1). Hint: write down the three relations among the coefficients that determine $\mathcal{SO}(2, \mathbb{R})$ and think of them as a smooth map $F : \mathbb{R}^4 \to \mathbb{R}^3$ that attains a specific value on $\mathcal{SO}(2, \mathbb{R})$; use the rank theorem.

2 Aufgabe

Consider the closed unit disc $D^2_1 \subset \mathbb{R}^2$ (as a topological space). On D^2_1 define an equivalence relation $(x, y) \sim (x', y')$ if $(x, y) = (x', y')$ or $x^2 + y^2 = x'^2 + y'^2 = 1$, i.e., we identify all the points on the boundary of the disc. Let $M := D^2_1/\sim$ be the quotient space and $\pi : D^2_1 \to M$ the quotient map.

(a) Show that M is Hausdorff and second countable (with its quotient topology).

(b) Define on M the structure of a smooth manifold diffeomorphic to S^2.

Suggested strategy: remember from last week the diffeomorphism $\phi : B^2_1 \to \mathbb{R}^2$ defined by

$$\phi((x_1, x_2)) = \left(\frac{x_1}{\sqrt{1 - x_1^2 - x_2^2}}, \frac{x_2}{\sqrt{1 - x_1^2 - x_2^2}}\right).$$

Use it to build the following atlas:

$$\tilde{\phi}_1 := \phi \circ \pi^{-1} : U_1 \to \mathbb{R}^2, \qquad U_1 := \pi(B^2_1),$$

$$\tilde{\phi}_2 := \sigma \circ \phi \circ \pi^{-1} : U_2 \to \mathbb{R}^2, \qquad U_2 := M \setminus \{\pi(0, 0)\}),$$

where σ is the map $\sigma : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2 \setminus \{(0, 0)\}$ given by $\sigma(x, y) := (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. (Note that σ is a diffeomorphism, and $\sigma^{-1} = \sigma$.) Strictly speaking, $\sigma \circ \phi \circ \pi^{-1}$ is defined only as map between $\pi(B^2_1 \setminus \{(0, 0)\})$ and $\mathbb{R}^2 \setminus \{(0, 0)\}$ but you can (and must!) extend it (uniquely) to a homeomorphism from $M \setminus \{\pi(0, 0)\}$ to $\mathbb{R}^2$. Explain why it is necessary and possible to do so. (Which point of M is mapped to $(0, 0)$?)

3 Aufgabe

Let M be a (smooth) n-manifold and $\{(\psi_\alpha, U_\alpha),\ \alpha \in I\}$ an atlas for M. Denote $\psi_\alpha(U_\alpha) =: V_\alpha \subset \mathbb{R}^n$. Consider the disjoint union $V := \coprod_{\alpha \in I} V_\alpha$. (If we give the index set I the discrete topology, this can be defined as $\{(x, \alpha) \in \mathbb{R}^n \times I \mid x \in V_\alpha\}$, but we usually write a point in V simply as $x_\alpha \in V_\alpha$.)

Define a relation on V as follows: For $x_\alpha \in V_\alpha$, $x_\beta \in V_\beta$, we say $x_\alpha \sim x_\beta$ if there exists $x \in U_\alpha \cap U_\beta$ such that $\psi_\alpha(x) = x_\alpha$ and $\psi_\beta(x) = x_\beta$.

(a) Show that this relation $\sim$ is an equivalence relation.

(b) Let $N := V/\sim$ denote the quotient space. Consider the map $\phi : V \to M$ defined by $\phi(x_\alpha) = \psi_\alpha^{-1}(x_\alpha)$ for $x_\alpha \in V_\alpha$. Show that this descends to a well-defined map $\bar{\phi} : N \to M$.

(c) Show that $\bar{\phi}$ is actually a homeomorphism. Use this homeomorphism to define a differentiable structure on N and conclude that M and N are diffeomorphic.