# Differentialgeometrie II

Test 2008 December 11

Solutions

## 1 Solution to the first problem

Choose coordinates $(x_1, y_1, \dots, x_n, y_n)$ and $(z_1, \dots, z_n)$ for $\mathbb{R}^{2n}$ and $\mathbb{C}^n$respectively. Then we can define an isomorphismus of the two spaces by sending $(x_1, y_1, \dots, x_n, y_n)$ to $(x_1 + iy_1, \dots, x_n + iy_n)$. In other words, $z_j = x_j + iy_j,\ j \in \{1, \dots, n\}$.

The action of the local flow $\theta_t(\vec{z}) = e^{it}\vec{z}$ viewed in $x_i, y_i$ coordinates is given by $\theta_t(x_1, y_1, \dots, x_n, y_n) = (\cos(t)x_1 - \sin(t)y_1,\ \cos(t)y_1 + \sin(t)x_1, \dots)$, i.e. $x_i$ is sent to $\cos(t)x_i - \sin(t)y_i$ and $y_i$ is sent to $\cos(t)y_i + \sin(t)x_i$.

We consider a generic $f \in C^\infty(\mathbb{R}^{2n})$. Then $X_{\vec{z}}f = \frac{d}{dt}\big|_{t=0} f \circ \theta_t(\vec{z}) = \sum_{i=1}^n ( -y_i \frac{\partial f}{\partial x_i}(\vec{z}) + x_i \frac{\partial f}{\partial y_i}(\vec{z}) )$. Thus $X = \sum_{i=1}^n ( -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} )$. This field is defined on the whole $\mathbb{R}^{2n}$. We must check that its restriction on the unit sphere $\mathbb{S}^{2n-1}$ actually belongs to the tangent space of the sphere.

Notice that the local flow leaves the sphere invariant, i.e. $\theta_t(\mathbb{S}^{2n-1}) = \mathbb{S}^{2n-1}$ : let $\vec{z} \in \mathbb{S}^{2n-1}, i.e. \|\vec{z}\|^2 = \sum_{i=1}^n \bar{z}_i z_i = 1$. Then $\|\theta_t(\vec{z})\| = \sum_{i=1}^n \bar{z}_i e^{-it} e^{it} z_i = 1$.

Thus each $\theta_t(\vec{z})$ with $\vec{z} \in \mathbb{S}^{2n-1}$ describes a curve on the sphere, and the associated velocity vectors belong to $T\mathbb{S}^{2n-1}$. So $\forall \vec{z} \in \mathbb{S}^{2n-1},\ X_{\vec{z}} \in T_{\vec{z}}\mathbb{S}^{2n-1}$.

Alternative strategy for proving the same fact: check that $X$ is orthogonal at each point $\vec{z} \in \mathbb{S}^{2n-1}$ to the normal field $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$.

To check that the field $X$ is nowhere zero on the sphere is trivial.

## 2 Solution to the second problem

- $\theta_t^*\Phi = \Phi, \forall t \in \mathbb{R}$ implies $L_X(\Phi) = 0$.

  This follows immediately, as $L_X(\Phi)$ has been defined as $\frac{d}{dt}\big|_{t=0}(\theta_t^*\Phi - \Phi)$ and $\theta_t$ leaves $\Phi$ invariant.

- $L_X(\Phi) = 0$ implies $\theta_t^*\Phi = \Phi, \forall t \in \mathbb{R}$.

  We know that $L_X(\Phi) = 0$, which means that for any $p \in M$, and any $X_p, Y_p \in T_pM$ we have $\frac{d}{dt}\big|_{t=0}\theta_t^*\Phi_{\theta_t(p)}(X_p, Y_p) = \frac{d}{dt}\big|_{t=0}\Phi_{\theta_t(p)}(\theta_{t*}(X_p), \theta_{t*}(Y_p)) = 0$. If we are able to establish the same kind of equation for any value of $t$ and not just for $t = 0$ we are done, as that would imply that $\theta_t^*\Phi$ is constant. So we have

$$\frac{d}{dt}\bigg|_{t=t_0} \Phi_{\theta_t(p)}(\theta_{t*}(X_p), \theta_{t*}(Y_p)) = \frac{d}{ds}\bigg|_{s=0} \Phi_{\theta_{s+t_0}}(p)(\theta_{s+t_0 *}(X_p), \theta_{s+t_0 *}(Y_p)).$$

Remembering that $\theta_{s+t_0} = \theta_s \circ \theta_{t_0}$ we can write the expression above as

$$\frac{d}{ds}\Big|_{s=0} \Phi_{\theta_s(\theta_{t_0}(p))}(\theta_{s*}(\theta_{t_0 *}(X_p)), \theta_{s*}(\theta_{t_0 *}(Y_p))).$$

But we already know that $\frac{d}{ds}\big|_{s=0} \Phi_{\theta_s(q)}(\theta_{s*}(X'_q), \theta_{s*}(Y'_q)) = 0$ for any $q \in M$ and for any $X'_q, Y'_q \in T_qM$, in particular for $q = \theta_{t_0}(p)$, $X'_q = \theta_{t_0 *}(X_p)$ and $Y'_q = \theta_{t_0 *}(Y_p)$.

# 3 Solution to the third problem

As stated, an inner product $\langle .,. \rangle$ on a finite dimensional vector space $V$ determines an isomorphism $\phi : V \to V^*$ onto its dual space: for $v \in V$, $\phi(v) \in V^*$ is defined by $\phi(v)(u) := \langle v, u \rangle, \forall u \in V$. $\phi(v)$ is an element of $V^*$ because of right linearity of the inner product. The map $\phi$ is a linear bijection because of left linearity and non-degeneracy of the inner product. To see this last point, choose a (non necessarily orthonormal) basis $\{e_i\}$ for $V$, denote with $\bar{e}_i$ the corresponding dual basis for $V^*$, i.e. $\bar{e}_i(e_j) = \delta_{i,j}$. Let $g_{i,j}$ be the metric matrix w.r.t. this basis, i.e. $g_{i,j} = \langle e_i, e_j \rangle$, and $g^{-1}_{i,j}$ its inverse matrix. Then $\phi^{-1}(w) = \sum_{i,j} w(e_i) g^{-1}_{i,j} e_j$. Likewise, $\phi(v) = \sum_{i,j} \bar{e}_i(v) g_{i,j} \bar{e}_j$. Check that these maps are mutually inverse and that they do not depend on the choice of basis.

This established, we know that for a Riemannian manifold $(M, \Phi)$ ($\Phi$ is here the metric form) for any point $p \in M$, $\Phi_p$ gives an isomorphism between the two spaces $T_pM$ and $T^*_pM$. Thus for any $C^\infty$ vector field $X$ we have a unique corresponding covector field $\Phi(X, .)$ and, vice-versa, for any $C^\infty$ covector field $\omega$ there is a unique corresponding vector field, which we will denote by $X_\omega$. One has to show that the corresponding fields (resp. covector fields) are $C^\infty$ as well.

Let $X$ be a $C^\infty$ vector field and $\omega$ a $C^\infty$ 1-form. Choose for an arbitrary $p \in M$ a coordinate neighbourhood $(U, \psi)$ with coordinate functions $x_i$ and associated frame $\{E_i\}$ and dual co-frame $\{\omega_i\}$. The coefficients of the metric w.r.t. this chart are $g_{i,j} = \Phi(E_i, E_j)$, where now $g_{i,j}$ are smooth functions defined on $U$. Likewise, denote with $g^{-1}_{i,j}$ the coefficient functions of the inverse. Then $\Phi(X, .) = \sum_{i,j} \omega_i(X) g_{i,j} \omega_j$. Analogously, $X_\omega = \sum_{i,j} \omega(E_i) g^{-1}_{i,j} E_j$. All the coefficient functions appearing in the above expressions are smooth.

If $f = x^2 - 2xy + z^3$, $gradf = (2x - 2y)\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y} + 3z^2\frac{\partial}{\partial z}$. It's zero on $(0, 0, 0)$.

# 4 Solution to problem four

For any $h \in C^\infty(M)$ we have

$$L_X(fY)h = [X, fY]h = Xf(Yh) - fY(Xh)$$

$$= (Xf)(Yh) + fX(Yh) - fY(Xh) = (Xf)(Yh) - f[X, Y]h$$

$$= (L_Xf)Y + fL_XY.$$