Differentialgeometrie II

Notes from 28.01.2009

1 Theorem. Any closed form $\omega \in \bigwedge^k(\mathbb{R}^n)$ is exact.

What follows is an alternative presentation of what you can find in *Boothby* lemma VI.7.11, lemma VI.7.12.

We must find $\phi \in \bigwedge^{k-1}(\mathbb{R}^n)$ such that $d\phi = \omega$. I will start describing ϕ indirectly (I will give an explicit expression below). All manifolds are understood to be oriented. We will use the stronger version of Stokes' theorem, valid for boundaries with edges and corners.

Let $\Gamma \subset \mathbb{R}^n$ be a $(k-1)$-dimensional submanifold (possibly with boundary). Join each point of Γ to the origin with a segment and build the cone $O\Gamma$, that is, the set $\{(tx^1, tx^2, \dots, tx^n)\}$ with $t \in [0,1], (x^1, \dots, x^n) \in \Gamma$. This is not, in general, a submanifold: try to picture some examples.

Consider the map $H : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ sending $(t, x^1, \dots, x^n)$ to $(tx^1, \dots, tx^n)$. Then $O\Gamma = H([0,1] \times \Gamma)$.

Now suppose we find a $(k-1)$-form ϕ such that, for any Γ as above, $\int_{O\Gamma} \omega = \int_\Gamma \phi$. Then we have the following

Lemma 1 For any k-dimensional submanifold B, we have $\int_B \omega = \int_B d\phi$.

Proof.

Consider the $(k+1)$ dimensional manifold $[0,1] \times B$. Its boundary is $(\{0\} \times B) \cup (\{1\} \times -B) \cup ([0,1] \times \partial B)$.

Beware of orientations: $-B$ denotes B with opposite orientation, and $[0,1]$ has the standard orientation.

$H^*\omega \in \bigwedge^k([0,1] \times \mathbb{R}^n)$ is a closed form, as $d \circ H^*\omega = H^* \circ d\omega = 0$. Thus we have

$$0 = \int_{[0,1]\times B} d \circ H^*\omega = \int_{\partial([0,1]\times B)} H^*\omega = \int_{\{0\}\times B} H^*\omega \; + \int_{\{1\}\times -B} H^*\omega \; + \int_{[0,1]\times \partial B} H^*\omega$$

$$= \int_{H(\{0\}\times B)} \omega \; + \int_{H(\{1\}\times -B)} \omega \; + \int_{H([0,1]\times\partial B)} \omega \; = - \int_B \omega + \int_{O\partial B} \omega$$

where we have used that $H(\{0\} \times B) = (0, \dots, 0)$ and $H(\{1\} \times \dots)$ is the identity map on $\mathbb{R}^n$.

But $\int_{O\partial B} \omega = \int_{\partial B} \phi = \int_B d\phi$, which proves the lemma. ∎

The fact that $d\phi = \omega$ follows from

Lemma 2 If for any k-dimensional submanifold B, $\int_B \omega = \int_B \omega'$, then $\omega = \omega'$.

The proof is left to you. It is basically the same as one of last week's exercises. ∎

So we have used a generalisation of the strategy we have seen in class for the case $\omega \in \bigwedge^1(\mathbb{R}^n)$ and ϕ a function (a "potential"). It remains that show that a $(k-1)$-form ϕ with the property $\int_{O\Gamma} \omega = \int_\Gamma \phi$ exists. Let $\omega = \sum_{1 \le i_1 < \dots < i_k \le n} a_{i_1,\dots,i_k}(x^1, \dots, x^n) dx^{i_1} \wedge \dots \wedge dx^{i_k}$. An explicit solution is

$$\phi = \sum_{1 \le i_1 < \dots < i_k \le n} \int_0^1 \sum_{\alpha=1}^k a_{i_1,\dots,i_k}(tx^1, \dots, tx^n) x^{i_\alpha} t^{k-1} (-1)^{\alpha-1} dt \; dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_\alpha}} \wedge \dots \wedge dx^{i_k}. \tag{1}$$

It is a good exercise to check that the above expression when integrated on a $(k-1)$-dimensional manifold Γ gives the integral of ω on the cone $O\Gamma$. Try it first for some simple low dimensional examples.

Remark The ϕ given in (1) is the explicit expression for $\mathcal{J}H^*\omega$ given in *Boothby*, VI.7.12.

To derive (alternatively) $d\phi = \omega$ by direct computation from (1) might be tricky.

Remark The proof holds for any "star-shaped" subset of $\mathbb{R}^n$.(cfr. definition in lemma VI.7.12)

Corollary 3 $H^i(\mathbb{R}^n) \cong 0$ *for* $i \geq 1$.

Remark You already know from *Boothby* theorem VI.7.8 that $H^0(\mathbb{R}^n) \cong \mathbb{R}$.

2 Example

Let $M = \mathbb{R}^2 \backslash \{0\}$. Let $\omega := d(\arctan(\frac{y}{x})) \in \bigwedge^1(M)$. ω is closed: this follows from $d \circ d = 0$. ω is not exact: for any closed (i.e. with no boundary) non self-intersecting curve γ, $\int_\gamma \omega = \pm 2\pi$ if γ goes round the origin (the sign depends on the orientation of γ), zero otherwise. This is obvious, especially if we think of $\omega = d\theta$ in polar coordinates. If ω were closed, then the value would have been zero in all cases (if $\omega = d\phi$ with $\phi \in \bigwedge^0(M)$, then for any curve $\gamma : [0,1] \to M$, $\int_\gamma \omega = \phi(\gamma(1)) - \phi(\gamma(0))$ and in the case the curve is closed $\gamma(1) = \gamma(0)$).

Note that whether we think of it as a multivalued function, a function on a Riemann surface or a function defined on M with a discontinuity on the y-axis, $\arctan(\frac{y}{x}) \notin \bigwedge^0(M)$. But there is no problem in considering $d\arctan(\frac{y}{x}) = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \bigwedge^1(M)$.

We cannot choose to integrate ω around the unit circle S^1 and invoke Stokes' theorem to conclude that the integral should be zero as the unit ball without the origin $B\backslash(0,0)$ is not compact.

You can also observe how the strategy of the theorem proved above breaks down: we cannot use the expression (1) as the domain of integration is not compact (in fact the integral diverges), nor can we choose another starting point for integration, as then some of the paths would meet the origin $(0,0)$.

3 Example

- $H^0(S^2) \cong \mathbb{R}$ follows from the fact that S^2 is connected.
- $H^1(S^2) \cong 0$ follows from the fact that any closed non self-intersecting curve γ in S^2 cuts the sphere into two disjoint parts. Thus $\gamma = \partial\Sigma$ is a boundary of a surface. So for any closed $\omega \in \bigwedge^1(S^2)$, $\int_\gamma \omega = \int_\Sigma dw = 0$. This implies that $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ for any two paths with same starting and ending points. Thus ω is exact(see problem 1).
- $H^2(S^2) \neq 0$. The volume form Ω is not exact (see theorem VI.7.9). Were $\Omega = d\phi$ with $\phi \in \bigwedge^1(S^2)$, then $\int_{S^2} \Omega = \int_{S^2} d\phi = \int_{\partial S^2} \phi = \int_\emptyset \phi = 0$.

4 Solution to one of the problems

The following was given as an optional problem some time ago.

Problem 4 *Let M be an n-dimensional manifold. Let $\omega \in \bigwedge^k(M)$. Show that $\omega = \sum_{\alpha=0}^{\infty} a_0^\alpha da_1^\alpha \wedge \cdots \wedge da_k^\alpha$, for some set of functions $a_i^\alpha \in \bigwedge^0(M)$. If M is compact, a finite set of functions can be chosen.*

Solution Use *Boothby* lemma V.4.1 to find a countable locally finite covering $\{(U_\beta, V_\beta)\}$ with $\bar{V}_\beta \subset U_\beta$ and local charts (U_β, Φ_β) with coordinates $x_\beta^1, \ldots, x_\beta^n$. For each of these charts we have $\omega_{|U_\beta} = \sum_{1 \leq i_1 < \ldots i_k \leq n} a_{i_1,\ldots,i_k} dx_\beta^{i_1} \wedge \cdots \wedge dx_\beta^{i_k}$ with $a_{i_1,\ldots,i_k} \in \bigwedge^0(U_\beta)$. Choose for each V_α a C^∞ function f_α with *suppf* $\subset \bar{U}_\beta$ and $f_\alpha = 1$ on V_α. Define now $\phi_\alpha := \sum_{1 \leq i_1 < \ldots i_k \leq n} f_\alpha a_{i_1,\ldots,i_k} \wedge d(f_\alpha x_\beta^{i_1}) \wedge \cdots \wedge d(f_\alpha x_\beta^{i_k})$ Extend all the functions appearing in this formula to functions in $\bigwedge^0(M)$ wich are zero out of U_β. We have $\omega_{|V_\alpha} = \phi_{\alpha|V_\alpha}$. Choose a partion of unity $\{h_\alpha\}$ subordinate to the covering $\{V_\alpha\}$ (striclty speaking, in order to use *Boothby* theorem V.4.4, you should first apply lemma V.4.1 again, this time to the covering $\{V_\alpha\}$). Then $\omega = \sum_\alpha h_\alpha \phi_\alpha$. If M is compact, we can choose a finite covering. ∎