1 Definitions of tensor fields

We met a generalisation of the concept of co-vectorfields: $C^\infty$-covariant tensor fields of order $r$ (Def. V.5.3).

These can be defined in the following three equivalent ways:

- a) a function $\Phi$ which assigns to each $p \in M$ an element $\Phi_p \in T^r(T_pM)$ and which has the following property: for any local chart $(U, \psi)$ with associated local frame $E_i$ the functions $\alpha_{i_1, \ldots, i_r} := \Phi(E_{i_1}, \ldots, E_{i_r})$ (the local coordinates for $\Phi$) belong to $C^\infty(U)$;
- b) a function $\Phi$ which assigns to each $p \in M$ an element $\Phi_p \in T^r(T_pM)$ and which has the property that for any choice $X_1, \ldots X_r$ of elements in $\mathcal{X}(M)$, $\Phi(X_1, \ldots X_r) \in C^\infty(M)$;
- b bis) a function $\Phi$ which assigns to each $p \in M$ an element $\Phi_p \in T^r(T_pM)$ and which has the property that for any open subset $U$ of $M$ and any choice $X_1, \ldots X_r$ of elements in $\mathcal{X}(U)$, $\Phi(X_1, \ldots X_r) \in C^\infty(U)$;
- c) a $C^\infty(M)$-linear function $\Phi$ from $\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)$ to $C^\infty(M)$.

Proof

a) $\Rightarrow$ b). We must show that for any $r$-tuple $r$ vector fields $X_1, \ldots , X_r$, $\Phi(X_1, \ldots , X_r) \in C^\infty(M)$. It is sufficient to show this locally for any chart $(U, \psi)$. Then the restrictions of the fields $X_i$ on $U$ are linear combinations $X_i = \sum \beta_i E_i$ with coefficients $\beta_i \in C^\infty(U)$. This observation plus a) imply that $\Phi(X_1, \ldots , X_r)|_U \in C^\infty(U)$. As this is true for any choice of $(U, \psi)$, b) is proven.

b) $\Rightarrow$ a). Let $(U, \psi)$ be a local chart with local frame $E_i$. We will show that for any $p \in U$ the functions $\alpha_{i_1, \ldots , i_r}$ are $C^\infty$ in a neighbourhood of $p$. Choose an open neighbourhood $V \ni p$ and a function $f$ such that $f = 1$ on $V$ and $\text{supp}(f) \subset U$. Then $fE_i \in \mathcal{X}(V)$ are global vector fields, zero on $M \setminus U$ and coinciding with $E_i$ on $V$. Thus $\alpha_{i_1, \ldots , i_r}|_V := \Phi(E_{i_1}, \ldots , E_{i_r})|_V = \Phi(fE_{i_1}, \ldots , fE_{i_r})|_V$. In the last expression all fields belong to $\mathcal{X}(M)$, therefore b) implies that $\alpha_{i_1, \ldots , i_r}$ is $C^\infty$ on $V$. 

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a) \Rightarrow b \ bis) \text{ Exactly as in } a) \Rightarrow b).

b) \Rightarrow a) \text{ Exactly as in } b) \Rightarrow a).

b) \Rightarrow c). \text{ Follows immediately by observing that each } \Phi_p \text{ is } \mathbb{R}\text{-multilinear. }

c) \Rightarrow b). \text{ The only non trivial fact we must show is that } \Phi \text{ assigns an element } 
\Phi_p \text{ in } T^*(T_pM) \text{ for each } p \in M, \text{ as this is not part of the hypothesis in } c).

For any } X_{1p}, \ldots, X_{rp} \in T_p(M) \text{ choose vector fields } X_i \in \mathcal{X}(M) \text{ such that } X_{i|p} = X_{ip}, \text{ i.e. the value of the fields at the point } p \text{ coincides with the above chosen vectors. It is always possible to find such vector fields. }

We must show that } \Phi_p \text{ is well defined: suppose we choose another set of vector fields } Y_1, \ldots, Y_r \text{ with the same property } (X_{ip} = Y_{i|p}), \text{ then we must have } 
\Phi(X_1, \ldots, X_r)(p) = \Phi(Y_1, \ldots, Y_r)(p). \text{ It is sufficient to prove the equality for one variable at a time, i.e. } 
\Phi(X_1, \ldots, X_i, \ldots, X_r)(p) = \Phi(X_1, \ldots, Y_i, \ldots, X_r)(p).

As a first step, we notice that if } X_{i|q} = Y_{i|q} \text{ not only for } q = p \text{ but for any } q \text{ in an open set } U \text{ containing } y, \text{ then the equality easily follows: we choose } 
f \in C^\infty(M) \text{ with } \text{supp}(f) \subset U \text{ and } f(p) = 1. \text{ Then } fX_i = fY_i. \text{ Thus } 
\Phi(X_1, \ldots, fX_i, \ldots, X_r)(p) = \Phi(X_1, \ldots, fY_i, \ldots, X_r)(p). \text{ As } \Phi \text{ is } C^\infty(M)\text{-linear by hypothesis, we have } 
f(p)\Phi(X_1, \ldots, X_i, \ldots, X_r)(p) = f(p)\Phi(X_1, \ldots, Y_i, \ldots, X_r)(p). \text{ As } f(p) = 1, \text{ the equality follows. }

Also notice that it is sufficient to show the following statement: for any } Y_i \text{ such that } Y_{i|p} = 0 \text{ it follows that } 
\Phi(X_1, \ldots, Y_i, \ldots, X_r)(p) = 0. \text{ The general statement follows by applying (multi) linearity to the field } X_i - Y_i.

Choose a local chart } (U, \psi) \text{ containing } p \text{ and choose a function } h \text{ and an open set } V \subset U \text{ containing } p \text{ such that } h = 1 \text{ on } V \text{ and } \text{supp}(h) \subset U. \text{ Let } Y_i \text{ be any vector field satisfying } Y_{i|p} = 0. \text{ In the open neighbourhood } U \text{ the restriction of } Y \text{ is equal to } \sum_i \beta_i E_i, \text{ where } E_i \text{ are the elements of the frame associated to the chart } (U, \psi). \text{ The functions } \beta_i \text{ satisfy } \beta_i(p) = 0. \text{ Consider the field } h^2Y. \text{ We have } Y_{i|V} = h^2Y_{i|V} = h^2\sum_i \beta_i E_i|_V. \text{ As } \text{supp}(h) \subset U, \text{ we extend } hE_i \text{ to vector fields defined on } M, \text{ by letting them be } 0 \text{ on } M \setminus U. \text{ Thus we have } 
\Phi(X_1, \ldots, Y_i, \ldots, X_r)(p) = \Phi(X_1, \ldots, h^2Y_i, \ldots, X_r)(p) 
= \Phi(X_1, \ldots, h^2\sum_i \beta_i E_i, \ldots, X_r)(p) = h(p)\sum_i \beta_i(p)\Phi(X_1, \ldots, hE_i, \ldots, X_r)(p).

These expressions are all zero, as we have } \beta_i(p) = 0.

Remark. \text{ The equation } \Phi(X_1, \ldots, h^2Y_i, \ldots, X_r)(p) = \Phi(X_1, \ldots, Y_i, \ldots, X_r)(p) \text{ follows from the partial result found above, as } h^2Y_i \text{ and } Y_i \text{ coincide not only on the point } p \text{ but on an open set.}

It would have been incorrect to say that as } Y_i = \sum_i \beta_i E_i \text{ and } \beta_i(p) = 0 \text{ it follows that } 
\Phi(X_1, \ldots, \sum_i \beta_i E_i, \ldots, X_r)(p) = \sum_i \beta_i(p)\Phi(X_1, \ldots, E_i, \ldots, X_r)(p) = 0, \text{ as the } E_i \text{ and } \beta_i \text{ are defined only on } U. \text{ We know that } \Phi \text{ is } C^\infty(M)\text{-linear, therefore we had to reduce the problem to one involving globally defined fields and functions.}
2 Some remarks on dual spaces

We saw that any finite dimensional (real) vector space $V$ is non canonically isomorphic to its dual space $V^*$: it is easy to describe an isomorphism once we have chosen a basis for $V$, and this isomorphism depends on the basis.

We can also consider the dual of the dual space $V^{**}$, i.e. the space of linear functionals on $V^*$. In this case there is a canonical isomorphism between $V$ and $V^{**}$. Let $v \in V$ and let $\bar{w} \in V^*$ be any element of the dual space. Define $\iota(v) \in V^{**}$ as the linear functional on $V^*$ given by the expression $\iota(v)(\bar{w}) := \bar{w}(v)$.

In other words, we can identify $V$ and $T^1(V^*)$, and use Def. V.6.1 and Theorem V.6.2, simply by replacing $V$ with $V^*$. Thus for $v, w \in V$ we can make sense of expressions like $v \otimes w$. And if $e_i$ is a basis for $V$, then $\{e_i \otimes \cdots \otimes e_i\}$ is a basis for $T^r(V^*)$.

Recalling Definition V.5.1, we see that $T^r(V^*)$ (the space of r-times linear functionals on $V^*$) is the same as $T_r(V)$ (the space of contravariant tensors of order $r$). More generally, a moment of thought shows that we have $T^r_s(V^*) = T^s_r(V)$.

We can extend Definition V.6.1 and theorem V.6.2 for the case of mixed covariant/contravariant tensors in a straightforward way. For example, let again $e_i$ be a basis for $V$ and let $\bar{e}_j$ be its dual basis. Then $\{\bar{e}_{j_1} \otimes \cdots \otimes \bar{e}_{j_s}, \otimes e_{i_1} \otimes \cdots \otimes e_{i_r}\}$ is a basis for $T^r_s(V)$.

All this carries on locally to the case of tensor fields. Let $M$ be a differentiable manifold and $(U, \psi)$ a local chart. Let $\{E_i\}$ be the associated local frame of vector fields and $\{\omega_i\}$ the associated coframe. Let $\Phi \in T^r_s(M)$ be a $C^\infty$ r-covariant s-contravariant tensor field on $M$. Then we have the following local description of our tensor field:

$$\Phi|_U = \sum_{i_1, \ldots, i_r, j_1, \ldots, j_s} \alpha_{i_1, \ldots, i_r, j_1, \ldots, j_s} \omega_{i_1} \otimes \cdots \otimes \omega_{i_r} \otimes E_{j_1} \otimes \cdots \otimes E_{j_s}$$

where the $\alpha_{i_1, \ldots, i_r, j_1, \ldots, j_s}$ belong to $C^\infty(U)$.

3 Example: Boothby, exercise V.5.1

Let $\{e_i\}$ be a basis for $V$ ans $\{\bar{e}_i\}$ the dual basis for $V^*$. For $r = 2$ $T^2(V)$, $\{\bar{e}_i \otimes \bar{e}_j\}$ is a basis (theorem V.6.2). Show that the following family of vectors off $T^2(V)$ is a basis as well: $\{\bar{e}_i \otimes \bar{e}_j + \bar{e}_j \otimes \bar{e}_i\}$ with $i \leq j$ and $\{\bar{e}_i \wedge \bar{e}_j\}$ with $i < j$. The first group of $\frac{n(n+1)}{2}$ vectors is a basis for $\Sigma^2(V)$ and the second group of $\frac{n(n-1)}{2}$ vectors is a basis for $\wedge^2(V)$.

Alternative proof Each element of $T^2(V)$ is sum of elements of the form $\bar{v} \otimes \bar{w}$, with $\bar{v}, \bar{w} \in T^1(V)$. But $\bar{v} \otimes \bar{w} = \frac{1}{2}((\bar{v} \otimes \bar{w} - \bar{w} \otimes \bar{v}) + \frac{1}{2}(\bar{v} \otimes \bar{w} + \bar{w} \otimes \bar{v}))$, i.e. a sum of an element of $\wedge^2(V)$ and an element of $\Sigma^2(V)$. In other words, $\bar{v} \otimes \bar{w} \in \wedge^2(V) \oplus \Sigma^2(V)$. This proves the assertion for $r = 2$. 

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Now consider the case $r > 2$. Consider the symmetrizing mapping $S : T^r(V) \to T^r(V)$ and the alternating mapping $A : T^r(V) \to T^r(V)$ introduced in Definition V.5.6. Check that their product is zero: $SA = AS = 0$.

Suppose that $T^r(V) = \Sigma^r(V) \oplus \Lambda^r(V)$. This means that any $\phi \in T^r(V)$ is of the form $\phi^+ + \phi^-$ with $\phi^+ \in \Sigma^r(V)$ (equivalently, $S\phi^+ = \phi^+$) and $\phi^- \in \Lambda^r(V)$ (equivalently, $A\phi^- = \phi^-$).

Consider the tensor $\phi := \sum_{\sigma} \bar{e}_1 \wedge \bar{e}_2 \otimes \bar{e}_{\sigma(3)} \otimes \cdots \otimes \bar{e}_{\sigma(r)}$, where the sum is over all permutations $\sigma$ involving only the last $r - 2$ terms and leaving the first two fixed. Check that $S\phi = 0$, which implies $\phi^+ = 0$. Likewise, check that $A\phi = 0$, which in turn implies $\phi^- = 0$. But $\phi$ is not zero, and we have just proven that $\phi \notin \Sigma^r(V) \oplus \Lambda^r(V)$.

### 4 Boothby, exercise V.5.2

We have seen that for any $\phi \in T^r(V)$ the following are equivalent:

- a) $\phi$ is antisymmetric;
- b) $\phi(v_1, \ldots, v_r) = 0$ whenever $v_i = v_j$ for $i \neq j$;
- c) $\phi(v_1, \ldots, v_r) = 0$ whenever $v_1, \ldots, v_r$ are linearly dependent.

a) $\Rightarrow$ b) follows form $\phi(v_1, \ldots, v_i, v_i, \ldots, v_r) = -\phi(v_1, \ldots, v_i, v_i, \ldots, v_r)$ (where we have permutated $v_i$ and $v_j$ = $v_i$).

b) $\Rightarrow$ c). Suppose that $v_1, \ldots, v_r$ are linearly dependent. Then, for example, $v_i = \sum_{j \neq i} \beta_j v_j$.

$\phi(v_1, \ldots, v_i, v_j, \ldots, v_r) = \sum_{j \neq i} \beta_j \phi(v_1, \ldots, v_j, v_i, \ldots, v_r) = 0$, as each $v_j$ appears (at least) twice.

c) $\Rightarrow$ b) obvious.

b) $\Rightarrow$ a). We know that $\phi(v_1, \ldots, v_i + v_j, \ldots, v_i + v_j, \ldots, v_r) = 0$. It follows, by $r$-linearity, that $\phi(v_1, \ldots, v_i, \ldots, v_r) + \phi(v_1, \ldots, v_j, \ldots, v_r) + \phi(v_1, \ldots, v_1, \ldots, v_r) + \phi(v_1, \ldots, v_j, \ldots, v_1, \ldots, v_r) = \phi(v_1, \ldots, v_i, v_j, \ldots, v_r) + \phi(v_1, \ldots, v_j, v_i, \ldots, v_r) = 0$. Thus $\phi$ is antisymmetric.

Now, when $r > \dim V$, it is obvious that any $r$-tuple of vectors is linearly dependent, so any antisymmetric $\phi \in \Lambda^r(V)$ must always take value zero.

### 5 An example using partitions of unity (Boothby, problem V.4.2)

Let $N$ be an $n$-dimensional closed regular submanifold of an $m$-dimensional manifold $M$. Show that a $C^\infty$ vector field $X$ on $N$ can be extend to a $C^\infty$-vector field on $M$. 

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As a first step, let’s prove the local version of this statement. Choose a point \( p \in N \) and a preferred (in the sense of Boothby Definition III.5.1) coordinate neighbourhood \((U, \phi)\) with coordinate functions \(x_1, \ldots, x_m\), and with \(E_1, \ldots, E_m\) the associated coordinate frame, such that \( q \in N \cap U \) iff the last \( m - n \) coordinate functions take value zero on \( q \). Denote by \((U \cap N, \tilde{\phi})\) the local chart for \( N \) given by the restriction \( \tilde{x}_1, \ldots, \) on \( U \cap N \) of first \( n \) coordinate functions \(x_1, \ldots, x_n\) (cfr. Boothby Lemma III.5.2.) Analogously, denote by \( \tilde{E}_1, \ldots, \tilde{E}_n \) the restrictions of the \( E_1, \ldots, E_n \).

Any smooth vector field \( X \) on \( U \cap N \) is of the form \( \sum_i \tilde{\alpha}_i \tilde{E}_i \), with \( \tilde{\alpha}_i \in C^\infty(U \cap N) \). In local coordinates these functions take the form \( \tilde{\alpha}_i \circ \tilde{\phi}^{-1} = \alpha_i(x_1, \ldots, x_n) \).

Consider the field \( X_U := \sum_i \alpha_i E_i \), where \( \alpha_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) := \tilde{\alpha}(x_1, \ldots, x_n) \). It is defined on all \( U \) and it is an extension of \( X \). This extension depends on the local chart.

Now let’s suppose we have a smooth field \( X \) defined on the whole of \( N \).

Do the same procedure as above for every point \( p \in N \). This gives a covering of \( N \). Add to this covering the set \( M \setminus N \), which is open as \( N \) is closed. Thus we obtain an open covering of \( M \). From Boothby Lemma V.4.1. we know that we can find a countable locally finite refinement \((U_i, V_i, \phi_i)\) which is regular (in the sense of remark 4.2). From Theorem V.4.4 we know that we can construct a smooth partition of unit \( f_i \) subordinate to this locally finite refinement. The idea is to paste together the local extensions \( X_{U_i} \) into a global extension using the functions \( f_i \), i.e. multiplying each \( X_{U_i} \) by \( f_i \) and extending its domain of definition to \( M \) by defying it to be zero out of \( U_i \) and finally considering the sum \( \sum_i f_i X_{U_i} \).

We have only a small problem. To pursue the above construction we need each \((U_i, V_i, \phi_i)\) to be a preferred neighbourhood, which is something which is not assured from Lemma V.4.1. To be precise, we have two kinds of neighbourhoods: those containing points of \( N \) (derived from the covering of \( N \)) and those which don’t (derived from \( M \setminus N \)). For these second kind of neighbourhoods simply define \( X_{U_i} \) to be zero.

We have two choices: go through the proof of the lemma, and see that when the original covering consists of subspaces associated to preferred charts, then it is possible to choose the locally finite refinement \( U_i, V_i, \phi_i \) with \( \phi \) preferred as well (I will not do this here, but it is not difficult). Second choice: remember that each \( U_i \) having non trivial intersection with \( N \) is contained in some \( U \) corresponding to a preferred chart \( U, \phi \) of the original covering. Simply forget the \( \phi_i \) given by the lemma, and define a new \( \phi_i \) as the restriction of the original \( \phi \) on \( U_i \). We are actually only interested in the existence of the \( f_i \), and we don’t need all the properties of a regular covering. Now we can pursue the construction stated above. Remark: the extension we found depends strongly on everything we chose: local charts, refinement, partitions of unity.

A counterexample which shows why we asked \( N \) to be closed. Consider \( M := \mathbb{R}^3 \), \( N := \{x, y, z \in \mathbb{R} \mid x^2 + y^2 = z^2\} \setminus \{(0, 0, 0)\} \), i.e. two cones without vertices.
embedded in $\mathbb{R}^3$. Let $X$ be, for example, $\frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ on the upper cone and zero on the lower one. This field cannot be extended, as we would have a singularity at the origin. In this example $N$ is not closed, so $M \setminus N$ is not part of an open covering of $M$ and the proof we gave above breaks down.