Differentialgeometrie II
Übungsblatt 1

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Warning: "coordinate neighbourhood","coordinate system", "(local) chart" are synonymous. I might switch from one terminology to the other - that will have no particular significance.

1 S^1

As a first example of a smooth (1-dimensional) manifold, let's consider the circle. Let $S^1 \subset \mathbb{R}^2$ the set of points $\{(x,y)\}$ of the cartesian plane satisfying $x^2+y^2=1$.

Choose a generic point $P=(x,y) \in S^1$ with $P \neq (0,1)$. Draw a straight line intersecting $(0,1)$ and P. This line will intersect the X-axis at the point $(\frac{x}{1-y}, 0)$.

Let's define the map $\psi_+ : U_+ \to \mathbb{R}$ as $\psi_+(x,y) = \frac{x}{1-y}$, where $U_+ = S^1 \backslash \{(0,1)\}$.

The inverse $\psi_+^{-1} : \mathbb{R} \to U_+$ is given by the expression $\psi_+^{-1}(X) = (\frac{2X}{1+X^2}, \frac{-1+X^2}{1+X^2})$, with $X \in \mathbb{R}$. Thus U_+ and $\mathbb{R}$ are isomorphic (as sets).

Furthermore both ψ_+ and ψ_+^{-1} are are easily seen to be continuous (usual topology for $\mathbb{R}$ and induced topology for S^1 as a subset of $\mathbb{R}^2$).

Thus ψ_+ establishes a homeomorphism between U_+ and $\mathbb{R}$, making the pair (U_+, ψ_+) a candidate for a coordinate neighbourhood.

Now consider the map $\sigma : S^1 \to S^1$ defined by $\sigma(x,y) = (-x,-y)$, which is a homeomorphism of the circle into itself.

Define $\psi_- : U_- \to \mathbb{R}$ as $\psi_- = \psi_+ \circ \sigma$, where $U_- = \sigma(U_+) = S^1 \backslash \{(0,-1)\}$. $\psi_-(x,y) = \frac{-x}{1+y}$.

The pair (U_-, ψ_-) is a candidate for a second coordinate neighbourhood. Notice that $U_+ \cup U_- = S^1$, so these two charts (if proven to be compatible) would be enough for establishing a smooth manifold structure on S^1.

Consider the composite $\psi_- \circ \psi_+^{-1} : \mathbb{R}\backslash 0 \to \mathbb{R}\backslash 0$. Notice that $0 \in \mathbb{R}$ is out of the domain of definition, as ψ_+^{-1} sends 0 to $(0,-1)$, and this is out of the domain of definition of ψ_-.

Composing the expressions for the two functions (or by easy geometric considerations) one finds that $\psi_- \circ \psi_+^{-1}(X) = -\frac{1}{X}$. This function (which is also its inverse) is easily seen to be smooth (remember that 0 is neither in the domain of definition nor in the image of the map).

Thus the two coordinate neighbourhood $(U_+, \psi_+), (U_-, \psi_-)$ define on S^1 a structure of C^∞ (i.e., smooth) 1-dimensional manifold.

2 S^n

We can apply the same procedure for the n-sphere $S^n \subset \mathbb{R}^{n+1}$ with coordinate neighbourhoods $(U_+, \psi_+), (U_n, \psi_n)$, where now $U_+ = S^n \backslash \{(0, 0, \dots 1)\}$, $\psi_+ : U_+ \to \mathbb{R}^n$, $\psi_+(x_1, x_2, \dots x_{n+1}) = (\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \dots \frac{x_n}{1-x_{n+1}})$ and U_- and ψ_- are defined analogously.

3 S^1 (again)

We look at S^1 from another point of view. Consider the set $\mathbb{R}/\mathbb{Z}$, i.e. the set of equivalence classes in $\mathbb{R}$ with respect to the equivalence relation $x \sim x + a$, $x \in \mathbb{R}, a \in \mathbb{Z}$. For $x \in \mathbb{R}$ we denote with $[x]$ the corresponding element in $\mathbb{R}/\mathbb{Z}$ and say that $x \in \mathbb{R}$ is a representative of the class $[x] \in \mathbb{R}/\mathbb{Z}$. In other words $[\ ] : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the quotient map, assigning to each real number its corresponding class. So, for example, $[x] = [x+3] = [x-111]$, where $x, x+3, \dots$ are different representatives of the same class. $\mathbb{Z}$ is the kernel of the map $[\ ]$.

Obviously every class, i.e. every element of $\mathbb{R}/\mathbb{Z}$ has one and only one representative in the interval $[0, 1)$ (we could have chosen any other half open half closed interval of unit length). That is, there is an isomorphism between the two sets $[0, 1)$ and $\mathbb{R}/\mathbb{Z}$.

Remark We endow $\mathbb{R}/\mathbb{Z}$ with the quotient topology[1] and we see that $[0, 1)$ and $\mathbb{R}/\mathbb{Z}$ are not isomorphic as topological spaces, i.e. not homeomorphic, as the latter is closed.

We want to give a structure of 1-dimensional smooth manifold to $\mathbb{R}/\mathbb{Z}$ and therefore look for coordinate neighbourhoods.

Consider the set $U_0 = [\ (0, 1)\ ] \subset \mathbb{R}/\mathbb{Z}$, i.e. the image in $\mathbb{R}/\mathbb{Z}$ through the quotient map $[\ ]$ of the set $(0, 1)$.

$[\ ]^{-1}(U_0) = \dots \cup (-1, 0) \cup (0, 1) \cup (1, 2) \dots$, i.e. a (countable) union of open sets, thus open. This means that $U_0 \subset \mathbb{R}/\mathbb{Z}$ is open as well (in the quotient topology).

Define $\phi_0 : U_0 \to (0, 1)$ as $\phi_0(m) = [\ ]^{(-1)}(m) \cap (0, 1)$, $\forall m \in U_0$.

That is, ϕ_0 assigns to each class contained in U_0 its unique representative in $(0, 1)$. This map is a bijection. It is also easily seen to be a homeomorphism. In fact, if $W \subset U_0$ is open, then $[\ ]^{-1}(W)$ must be an open subset of $\mathbb{R}$, and the intersection of two open sets $[\ ]^{-1}(W) \cap (0, 1)$ is open as well. Likewise, if $V \subset (0, 1)$ is open, then $\phi_0^{-1}(V) = [V] \subset \mathbb{R}/\mathbb{Z}$ is open (proof analogous as for U_0 above).

So (U_0, ϕ_0) is a candidate for a coordinate neighbourhood.

[1] a set $U \subset \mathbb{R}/\mathbb{Z}$ is said to be open in the quotient topology iff $[\ ]^{-1}(U)$ is open in $\mathbb{R}$.

We construct a second (candidate) chart by shifting the same procedure 1/2 on the right. Define: $U_{\frac{1}{2}} := [\ (\frac{1}{2}, \frac{3}{2})\ ]$ and $\phi_{\frac{1}{2}} : U_{\frac{1}{2}} \to (\frac{1}{2}, \frac{3}{2})$ as $\phi_{\frac{1}{2}}(m) := [\ ]^{-1}(m) \cap (\frac{1}{2}, \frac{3}{2})$.

U_0 and $U_{\frac{1}{2}}$ are a covering of $\mathbb{R}/\mathbb{Z}$, i.e. $U_0 \cup U_{\frac{1}{2}} = \mathbb{R}/\mathbb{Z}$.

$U_0 \cap U_{\frac{1}{2}} = (\mathbb{R}/\mathbb{Z})\backslash\{[0], [\frac{1}{2}]\}$, i.e. their intersection leaves two points out. Notice also that this intersection is not connected.

Likewise, the images in $\mathbb{R}$ of this intersection given by ϕ_0 and $\phi_{\frac{1}{2}}$ are disconnected: $\phi_0(U_0 \cap U_{\frac{1}{2}}) = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $\phi_{\frac{1}{2}}(U_0 \cap U_{\frac{1}{2}}) = (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$.

So we consider now the composition $\phi_{\frac{1}{2}} \circ \phi_0^{-1} : (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \to (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$.

This map is a diffeomorphism: $\phi_{\frac{1}{2}} \circ \phi_0^{-1}|_{(\frac{1}{2},1)} : (\frac{1}{2}, 1) \to (\frac{1}{2}, 1)$ is the identity map and $\phi_{\frac{1}{2}} \circ \phi_0^{-1}|_{(0,\frac{1}{2})} : (0, \frac{1}{2}) \to (1, \frac{3}{2})$ is simply the map $x \to x + 1$.

This shows that the two coordinate neighbourhoods $(U_0, \phi_0), (U_{\frac{1}{2}}, \phi_{\frac{1}{2}})$ determine a smooth structure on $\mathbb{R}/\mathbb{Z}$.

Remark It is possible to construct a diffeomorphism between S^1 in the first example and $\mathbb{R}/\mathbb{Z}$. For example, assign to each point $P \in S^1$ with coordinates (x, y) the angle θ such that $cos(\theta) = x,\ sin(\theta) = y$. Then the map $\pi : S^1 \to \mathbb{R}/\mathbb{Z}, \pi(x, y) = \frac{1}{2\pi}\theta$ is an isomorphism (of sets). Looking at the action of this map in terms of local charts, i.e. studying the functions $\phi_j \circ \pi \circ \psi_i^{-1},\ (i = +, -;\ j = 0, \frac{1}{2})$ one sees that it is actually a diffeomorphism.

In other words, S^1 and $\mathbb{R}/\mathbb{Z}$ (with the smooth structures given above) are diffeomorphically equivalent.

Remark We were able to define a smooth 1-dimensional manifold structure for $\mathbb{R}/\mathbb{Z}$ without mentioning $\mathbb{R}^2$.

More precisely, in the first example we have defined S^1 as a 1-dimensional smooth manifold embedded in $\mathbb{R}^2$, while in the third example we have constructed a (diffeomorphically equivalent) 1-dimensional manifold directly, without the help auxiliary higher dimensional manifolds.

4 T^2

As in the previous example, consider the 2-dimensional torus $\mathbb{R}^2\backslash\mathbb{Z}^2$. One can choose open squares of the type $(a, a + 1) \times (b, b + 1) \subset \mathbb{R}^2$ and define charts $(U_{a,b}, \phi_{a,b})$ as before:

$U_{a,b} := [(a, a + 1) \times (b, b + 1)] \subset \mathbb{R}^2\backslash\mathbb{Z}^2$;

$\phi_{a,b}(m) := [\ ]^{-1}(m) \cap ((a, a + 1) \times (b, b + 1))$.

A choice of charts of such kind necessary for determining a smooth structure on the torus is left to the reader.