Differentialgeometrie II
Übungsblatt 1

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Warning: “coordinate neighbourhood”, ”coordinate system”, “(local) chart” are synonymous. I might switch from one terminology to the other - that will have no particular significance.

1 $S^1$

As a first example of a smooth (1-dimensional) manifold, let’s consider the circle. Let $S^1 \subset \mathbb{R}^2$ the set of points \{(x, y)\} of the cartesian plane satisfying $x^2 + y^2 = 1$.

Choose a generic point $P = (x, y) \in S^1$ with $P \neq (0, 1)$. Draw a straight line intersecting $(0, 1)$ and $P$. This line will intersect the X-axis at the point \((\frac{x}{1-y}, 0)\).

Let’s define the map $\psi_+ : U_+ \to \mathbb{R}$ as $\psi_+(x, y) = \frac{x}{1-y}$, where $U_+ = S^1 \setminus \{(0, 1)\}$.

The inverse $\psi_+^{-1} : \mathbb{R} \to U_+$ is given by the expression $\psi_+^{-1}(X) = (\frac{2X}{1+X^2}, \frac{1+X^2}{1+X^2})$, with $X \in \mathbb{R}$. Thus $U_+$ and $\mathbb{R}$ are isomorphic (as sets).

Furthermore both $\psi_+$ and $\psi_+^{-1}$ are easily seen to be continuous (usual topology for $\mathbb{R}$ and induced topology for $S^1$ as a subset of $\mathbb{R}^2$).

Thus $\psi_+$ establishes a homeomorphism between $U_+$ and $\mathbb{R}$, making the pair $(U_+, \psi_+)$ a candidate for a coordinate neighbourhood.

Define $\psi_- : U_- \to \mathbb{R}$ as $\psi_- = \psi_+ \circ \sigma$, where $U_- = \sigma(U_+) = S^1 \setminus \{(0, -1)\}$.

$\psi_-(x, y) = \frac{-x}{1+y}$.

The pair $(U_-, \psi_-)$ is a candidate for a second coordinate neighbourhood.

Notice that $U_+ \cup U_- = S^1$, so these two charts (if proven to be compatible) would be enough for establishing a smooth manifold structure on $S^1$.

Consider the composite $\psi_- \circ \psi_+^{-1} : \mathbb{R} \setminus 0 \to \mathbb{R} \setminus 0$. Notice that $0 \in \mathbb{R}$ is out of the domain of definition, as $\psi_+^{-1}$ sends 0 to $(0, -1)$, and this is out of the domain of definition of $\psi_-$. Composing the expressions for the two functions (or by easy geometric considerations) one finds that $\psi_- \circ \psi_+^{-1}(X) = -\frac{x}{y}$. This function (which is also its
inverse) is easily seen to be smooth (remember that 0 is neither in the domain of definition nor in the image of the map).

Thus the two coordinate neighbourhood \((U_+, \psi_+), (U-, \psi_-)\) define on \(S^1\) a structure of \(C^\infty\) (i.e., smooth) 1-dimensional manifold.

### 2 \(S^n\)

We can apply the same procedure for the n-sphere \(S^n \subset \mathbb{R}^{n+1}\) with coordinate neighbourhoods \((U_+, \psi_+), (U_n, \psi_n)\), where now \(U_+ = S^n \setminus \{(0, 0, \ldots, 1)\}\), \(\psi_+ : U_+ \to \mathbb{R}^n\), \(\psi_+(x_1, x_2, \ldots, x_{n+1}) = \left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right)\) and \(U_-\) and \(\psi_-\) are defined analogously.

### 3 \(S^1\) (again)

We look at \(S^1\) from another point of view. Consider the set \(\mathbb{R}/\mathbb{Z}\), i.e. the set of equivalence classes in \(\mathbb{R}\) with respect to the equivalence relation \(x \sim x + a, x \in \mathbb{R}, a \in \mathbb{Z}\). For \(x \in \mathbb{R}\) we denote with \([x]\) the corresponding element in \(\mathbb{R}/\mathbb{Z}\) and say that \(x \in \mathbb{R}\) is a representative of the class \([x]\) \(\in \mathbb{R}/\mathbb{Z}\). In other words \([\cdot] : \mathbb{R} \to \mathbb{R}/\mathbb{Z}\) is the quotient map, assigning to each real number its corresponding class. So, for example, \([x] = [x+3] = [x-11]\), where \(x, x+3, \ldots\) are different representatives of the same class. \(\mathbb{Z}\) is the kernel of the map \([\cdot]\).

Obviousely every class, i.e. every element of \(\mathbb{R}/\mathbb{Z}\) has one and only one representative in the interval \([0, 1)\) (we could have chosen any other half open half closed interval of unit length). That is, there is an isomorphism between the two sets \([0, 1)\) and \(\mathbb{R}/\mathbb{Z}\).

**Remark** We endow \(\mathbb{R}/\mathbb{Z}\) with the quotient topology\(^1\) and we see that \([0, 1)\) and \(\mathbb{R}/\mathbb{Z}\) are not isomorphic as topological spaces, i.e. not homeomorphic, as the latter is closed.

We want to give a structure of 1-dimensional smooth manifold to \(\mathbb{R}/\mathbb{Z}\) and therefore look for coordinate neighbourhoods.

Consider the set \(U_0 = \{\{(0, 1)\}\} \subset \mathbb{R}/\mathbb{Z}\), i.e. the image in \(\mathbb{R}/\mathbb{Z}\) through the quotient map \([\cdot]\) of the set \((0, 1)\).

\([\cdot]^{-1}(U_0) = \cdots \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cdots\), i.e. a (countable) union of open sets, thus open. This means that \(U_0 \subset \mathbb{R}/\mathbb{Z}\) is open as well (in the quotient topology).

Define \(\phi_0 : U_0 \to (0, 1)\) as \(\phi_0(m) = [\cdot]^{-1}(m) \cap (0, 1), \forall m \in U_0\).

That is, \(\phi_0\) assigns to each class contained in \(U_0\) its unique representative in \((0, 1)\). This map is a bijection. It is also easily seen to be a homeomorphism. In fact, if \(W \subset U_0\) is open, then \([\cdot]^{-1}(W)\) must be an open subset of \(\mathbb{R}\), and the intersection of two open sets \([\cdot]^{-1}(W) \cap (0, 1)\) is open as well. Likewise, if \(V \subset (0, 1)\) is open, then \(\phi_0^{-1}(V) = [V] \subset \mathbb{R}/\mathbb{Z}\) is open (proof analogous as for \(U_0\) above).

So \((U_0, \phi_0)\) is a candidate for a coordinate neighbourhood.

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\(^1\) a set \(U \subset \mathbb{R}/\mathbb{Z}\) is said to be open in the quotient topology iff \([\cdot]^{-1}(U)\) is open in \(\mathbb{R}\).
We construct a second (candidate) chart by shifting the same procedure 1/2
on the right. Define: \( U_2 := [(\frac{1}{2}, \frac{3}{2})] \) and \( \phi_2 : U_2 \to (\frac{1}{2}, \frac{3}{2}) \) as \( \phi_2(m) := \lfloor m \rfloor \mod 1 \).

\( U_0 \) and \( U_\frac{1}{2} \) are a covering of \( \mathbb{R}/\mathbb{Z} \), i.e. \( U_0 \cup U_\frac{1}{2} = \mathbb{R}/\mathbb{Z} \).

\( U_0 \cap U_\frac{1}{2} = (\mathbb{R}/\mathbb{Z}) \setminus \{[0], [\frac{1}{2}]\} \), i.e. their intersection leaves two points out. Notice also that this intersection is not connected.

Likewise, the images in \( \mathbb{R} \) of this intersection given by \( \phi_0 \) and \( \phi_\frac{1}{2} \) are disconnected: \( \phi_0(U_0 \cap U_\frac{1}{2}) = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \) and \( \phi_\frac{1}{2}(U_0 \cap U_\frac{1}{2}) = (\frac{1}{2}, 1) \cup (1, \frac{3}{2}) \).

So we consider now the composition \( \phi_\frac{1}{2} \circ \phi_0^{-1} : (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \to (\frac{1}{2}, 1) \cup (1, \frac{3}{2}) \).

This map is a diffeomorphism: \( \phi_\frac{1}{2} \circ \phi_0^{-1}|_{(0, \frac{1}{2})} : (0, \frac{1}{2}) \to (\frac{1}{2}, 1) \) is simply the map \( x \to x + 1 \).

This shows that the two coordinate neighbourhoods \( (U_0, \phi_0), (U_\frac{1}{2}, \phi_\frac{1}{2}) \) determine a smooth structure on \( \mathbb{R}/\mathbb{Z} \).

Remark It is possible to construct a diffeomorphism between \( S^1 \) in the first example and \( \mathbb{R}/\mathbb{Z} \). For example, assign to each point \( P \in S^1 \) with coordinates \( (x, y) \) the angle \( \theta \) such that \( \cos(\theta) = x, \sin(\theta) = y \). Then the map \( \pi : S^1 \to \mathbb{R}/\mathbb{Z}, \pi(x, y) = \frac{1}{2\pi} \theta \) is an isomorphism (of sets). Looking at the action of this map in terms of local charts, i.e. studying the functions \( \phi_j \circ \pi \circ \psi_i^{-1} \), \( i = +, -; j = 0, \frac{1}{2} \)

one sees that it is actually a diffeomorphism.

In other words, \( S^1 \) and \( \mathbb{R}/\mathbb{Z} \) (with the smooth structures given above) are diffeomorphically equivalent.

Remark We were able to define a smooth 1-dimensional manifold structure for \( \mathbb{R}/\mathbb{Z} \) without mentioning \( \mathbb{R}^2 \).

More precisely, in the first example we have defined \( S^1 \) as a 1-dimensional smooth manifold embedded in \( \mathbb{R}^2 \), while in the third example we have constructed a (diffeomorphically equivalent) 1-dimensional manifold directly, without the help auxiliary higher dimensional manifolds.

4 \( T^2 \)

As in the previous example, consider the 2-dimensional torus \( \mathbb{R}^2 \setminus \mathbb{Z}^2 \). One can choose open squares of the type \( (a, a + 1) \times (b, b + 1) \subset \mathbb{R}^2 \) and define charts \( (U_{a,b}, \phi_{a,b}) \) as before:

\[ U_{a,b} := [(a, a + 1) \times (b, b + 1)] \subset \mathbb{R}^2 \setminus \mathbb{Z}^2; \]

\[ \phi_{a,b}(m) := \lfloor m \rfloor \mod 1 \cap ((a, a + 1) \times (b, b + 1)). \]

A choice of charts of such kind necessary for determining a smooth structure on the torus is left to the reader.