Differential Geometry II

John M. Sullivan

Final Exam, due 2009 February 19 by 15:00 in MA 320

You must work on these four problems on your own; do not discuss them with other people. You may use books, notes and other written sources (like the web). If you do use material from such sources (other than Boothby's book), please cite your sources. If you have questions please contact me (for instance by email); to repeat, you may not ask questions to anyone else. Please sign the statement at the bottom of the test to confirm that your work is your own.

Note that throughout this exam "manifold" means manifold without boundary.

1. Mapping degree. Suppose M and N are connected, compact, oriented manifolds of the same dimension n. Define the degree $\deg F$ of a smooth map $F\colon M \to N$ as follows. Take any n-form ω on N with $\int_N \omega \neq 0$, and set

$$\deg F := \frac{\int_M F^*\omega}{\int_N \omega}.$$

Show that $\deg F$ is well-defined, that is, that it does not depend on the choice of ω. (Hint: assume the theorem that $H^n(N^n) \cong \mathbb{R}$ with the isomorphism given by integration.)

Note: in fact, $\deg F$ is always an integer. This is most easily proved by considering alternative definitions of cohomology. It implies that continuous changes to the map F leave the degree unchanged.

2. Uniqueness of isometries. Any orientation-preserving isometry of $\mathbb{R}^n$ is a product of a translation and a rotation. Most other Riemannian manifolds have many fewer isometries. However, the isometries that do exist are still determined uniquely by a translation and a rotation in the following sense.

Let M be a connected Riemannian manifold, and let p and q be points in M. Suppose f and g are two isometries $M \to M$ such that $f(p) = q = g(p)$ and $f_* = g_* : T_pM \to T_qM$. Prove that $f = g$. (Hint: use normal coordinates to show first that this is true in a neighborhood of p.)

3. Existence of self-diffeomorphisms.

(a) Let (ϕ, U) be a chart on a manifold M, with $\phi(U) = \mathbb{R}^m$. If p and q are any points in U, show there is a diffeomorphism $F\colon M \to M$ with $F(p) = q$.

(b) If p and q are any two points of a connected manifold M, show there is a diffeomorphism of M taking p to q.

4. Bicycles, planimeters and automobiles. Consider a rod of unit length in the plane $\mathbb{R}^2$. One end (the "pointer" or front wheel) is free to move to any point $(x, y) \in \mathbb{R}^2$. The other end (the "slider" or rear wheel) must follow at distance one. The configuration is described by $(x, y, \theta) \in \mathbb{R}^2 \times S^1$, where the pointer is at (x, y) and the slider at $(\tilde{x}, \tilde{y}) = (x + \cos\theta, y + \sin\theta)$.

The pointer can move arbitrarily, but the slider always follows along at distance one, and its motion is always along the line connecting the two points. We can express this condition by saying that the motion of the rod in its phase space $\mathbb{R}^2 \times S^1$ is always tangent to the planes spanned by two particular vectorfields

$$X = \partial_x + f\partial_\theta, \quad Y = \partial_y + g\partial_\theta.$$

(a) Find the functions f and g in the expression above, and then compute the Lie bracket $[X, Y]$. Is the 2-plane field spanned by X and Y involutive? (Hint: you might first consider the effect that moving the pointer has on the position $(\tilde{x}, \tilde{y})$ of the slider, then translate this to θ.)

(b) If we move the pointer around a closed loop, when we return to the starting (x, y), the rod will have twisted slightly, that is, the angle θ does not return to its initial value. Interpret the change $\Delta\theta$ in terms of $[X, Y]$ if the loop is a small Δx by Δy rectangle in the plane. Explain how the rod functions as a "planimeter", that is, how it approximately measures the area enclosed when we move the pointer around any loop.

Note that when parallel parking a car, one must make use of this Lie bracket $[X, Y]$. Only two dimensions of control are available directly, but luckily their bracket gives the third direction of motion in phase space.

With my signature below, I confirm that my work on this exam is mine alone, that I have not discussed these problems with other people, and that I have acknowledged any use of written sources other than Boothby.