Differentialgeometrie II
Übungsblatt 4 - Lösungen

December 7, 2008

1 Aufgabe

Let $M = \mathbb{R}^2$ and $\theta : \mathbb{R} \times M \to M$ be given by the formula

$$\theta_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t).$$

- Show that $\theta$ is a globally defined action of $\mathbb{R}$ on $M$.
- Describe $X$, the associated infinitesimal generator.
- Describe the orbits.
- Show that $X$ is invariant with respect to $\theta$, i.e. $\theta_t(X(x,y)) = X(\theta_t(x,y))$.

Solution

- It is pretty evident that $\theta$ is defined for any $t \in \mathbb{R}$, so it is globally defined (by the way, each $\theta_t$ is a diffeomorphism from $M$ onto $M$). Matrix multiplication and well known properties of $\cos(t)$ and $\sin(t)$ show that $\theta_t$ gives an action of $\mathbb{R}$ on $M$, i.e. $\theta_t \circ \theta_s = \theta_{t+s}$ and $\theta_0 = \text{id}$.
- Let $p \in M$ and $f \in C^\infty(p)$. We write $f(x,y)$, where $x, y$ are usual coordinates for $\mathbb{R}^2$ and $p = (x, y)$. Then $X_p f = \frac{\partial}{\partial t} f \circ \theta_t(x,y)|_{t=0} = \frac{\partial}{\partial t} f(x \cos t + y \sin t, -x \sin t + y \cos t)|_{t=0} = y \frac{\partial}{\partial y} f(x, y) - x \frac{\partial}{\partial y} f(x, y)$. Thus $X_p = y(\frac{\partial}{\partial x})|_p - x(\frac{\partial}{\partial y})|_p$.
- The flow consists of regular circular motions around the origin (clockwise), thus the orbits are circles.
- Let $f \in C^\infty(\theta_t(p))$, $p = (x, y)$.

$$X_{\theta_t(x,y)} f = (-x \sin(t) + y \cos(t)) \frac{\partial}{\partial x} f|_{\theta_t(x,y)} - (x \cos(t) + y \sin(t)) \frac{\partial}{\partial y} f|_{\theta_t(x,y)}.$$ 

Thus $\theta_t(X_{(x,y)}) = X_{(x+y)} f \circ \theta_t(x,y) = X_{(x,y)} f(x \cos t + y \sin t, -x \sin t + y \cos t) = y \frac{\partial}{\partial y} f(x \cos t + y \sin t, -x \sin t + y \cos t)|_p - x \frac{\partial}{\partial y} f(x \cos t + y \sin t, -x \sin t + y \cos t)|_p = (-x \sin(t) + y \cos(t)) \frac{\partial}{\partial x} f|_{\theta_t(x,y)} - (x \cos(t) + y \sin(t)) \frac{\partial}{\partial y} f|_{\theta_t(x,y)}$.

2 Aufgabe

Let $M = \mathbb{R}^2$, the $x, y$ plane, and $X = y(\frac{\partial}{\partial x}) + x(\frac{\partial}{\partial y})$. Find the corresponding domain $W$ and one-parameter group $\theta : W \to M$.

Solution Let $\theta_t(x_0, y_0) = (x(t), y(t))$. Then $\dot{x}(t) = y(t)$ and $\dot{y}(t) = x(t)$. The solution is given by $(x(t), y(t)) = (x_0 \cosh(t) + y_0 \sinh(t), x_0 \sinh(t) + y_0 \cosh(t))$, which is defined for any $t \in \mathbb{R}$. Thus $W = \mathbb{R} \times M$. 

1
3 Aufgabe

Let $X = x^2 \frac{\partial}{\partial x}$ be a vector field on $M = \mathbb{R}$. Find the associated flow $\theta$ and describe its domain $W$.

Solution

Let $\theta_t(x_0) = x(t)$. We must have (where defined) $\dot{x}(t) \frac{\partial}{\partial x} = x^2 \frac{\partial}{\partial x}$, so we obtain the differential equation $\dot{x}(t) = x^2(t)$. The solution is $\theta_t(x_0) = \frac{x_0}{1-tx_0}$. We have three cases:

- $x_0 = 0$, $I_{x_0} = \mathbb{R}$; orbit $\{0\}$;
- $x_0 > 0$, $I_{x_0} = (-\infty, 0]$; orbit $(0, \infty)$;
- $x_0 < 0$, $I_{x_0} = (0, \infty)$; orbit $(-\infty, 0)$.

$W = \bigsqcup_{x_0 \in \mathbb{R}} I_{x_0}$.

Remark

We can view $\theta_t(x_0) = x(t)$ as a map from $I_{x_0}$ to the orbit of $x_0$, a subset of $\mathbb{R}$. Then $\theta_*(x_0)\left(\frac{\partial}{\partial t}\right) = x^2 \frac{\partial}{\partial x}$ is an equation between vector fields. Also, if we view $t(x)$ as a function on the orbit of $x_0$ (the inverse function of $x(t)$), then we can interpret $dt = x^{-2}dx$ as an equation between 1-forms.

4 Aufgabe

Let $M = GL(2, \mathbb{R})$ and define an action of $\mathbb{R}$ on $M$ by the formula

$$\theta(t, A) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot A, \quad A \in GL(2, \mathbb{R}),$$

with the dot denoting matrix multiplication. Find the infinitesimal generator.

Solution Let’s choose coordinates $\{x_i\}$ to represent a generic element $A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in GL(2, \mathbb{R})$.

Let $f(x_1, x_2, x_3, x_4)$ denote an element in $C^\infty(A)$, then

$$f \circ \theta_t(x_1, x_2, x_3, x_4) = f(x_1 + tx_3, x_2 + tx_4, x_3, x_4).$$

$$X_A f = \frac{d}{dt} f(x_1 + tx_3, x_2 + tx_4, x_3, x_4)_{|t=0} = x_3 \frac{\partial}{\partial x_1} f_{|A} + x_4 \frac{\partial}{\partial x_2} f_{|A}.$$ 

Thus $X = x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2}$. 

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