Differentialgeometrie II
Übungsblatt 4 - Lösungen

December 7, 2008

1 Aufgabe

Let $M = \mathbb{R}^2$ and $\theta : \mathbb{R} \times M \to M$ be given by the formula

$$\theta_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t).$$

- Show that θ is a globally defined action of $\mathbb{R}$ on M.
- Describe X, the associated infinitesimal generator.
- Describe the orbits.
- Show that X is invariant with respect to θ, i.e. $\theta_{t*}(X_{(x,y)}) = X_{\theta_t(x,y)}$.

Solution

- It is pretty evident that θ is defined for any $t \in \mathbb{R}$, so it is globally defined (by the way, each θ_t is a diffeomorphism from M onto M). Matrix multiplication and well known properties of $\cos(t)$ and $\sin(t)$ show that θ_t gives an action of $\mathbb{R}$ on M, i.e. $\theta_t \circ \theta_s = \theta_{t+s}$ and $\theta_0 = id$.
- Let $p \in M$ and $f \in C^\infty(p)$. We write $f(x, y)$, where x, y are usual coordinates for $\mathbb{R}^2$ and $p = (x, y)$. Then $X_p f = \frac{d}{dt} f \circ \theta_t(x, y)_{|_{t=o}} = \frac{d}{dt} f(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t))_{|_{t=o}} = y \frac{\partial}{\partial x} f(x, y) - x \frac{\partial}{\partial y} f(x, y)$. Thus $X_p = y(\frac{\partial}{\partial x})_{|_p} - x(\frac{\partial}{\partial y})_{|_p}$.
- The flow consists of regular circular motions around the origin (clockwise), thus the orbits are circles..
- Let $f \in C^\infty(\theta_t(p)),\ p = (x, y)$.
  $X_{\theta_t(x,y)} f = (-x \sin(t) + y \cos(t)) \frac{\partial}{\partial x} f_{|_{\theta_t(p)}} - (x \cos(t) + y \sin(t)) \frac{\partial}{\partial y} f_{|_{\theta)t(p)}}$.
  $\theta_{t*}(X_{(x,y)}) f = X_{(x,y)} f \circ \theta_t(x, y) = X_{(x,y)} f(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t)) =$
  $y \frac{\partial}{\partial x} f(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t))_{|_p} - x \frac{\partial}{\partial y} f(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t))_{|_p} =$
  $(-x \sin(t) + y \cos(t)) \frac{\partial}{\partial x} f_{|_{\theta_t(p)}} - (x \cos(t) + y \sin(t)) \frac{\partial}{\partial y} f_{|_{\theta)t(p)}}$.

2 Aufgabe

Let $M = \mathbb{R}^2$, the x, y plane, and $X = y(\frac{\partial}{\partial x}) + x(\frac{\partial}{\partial y})$. Find the corresponding domain W and one-parameter group $\theta : W \to M$.

Solution Let $\theta_t(x_0, y_0) = (x(t), y(t))$. Then $\dot{x}(t) = y(t)$ and $\dot{y}(t) = x(t)$. The solution is given by $(x(t), y(t)) = (x_0 \cosh(t) + y_0 \sinh(t), x_0 \sinh(t) + y_0 \cosh(t))$, which is defined for any $t \in \mathbb{R}$. Thus $W = \mathbb{R} \times M$.

3 Aufgabe

Let $X = x^2 \frac{\partial}{\partial x}$ be a vector field on $M = \mathbb{R}$. Find the associated flow θ and describe its domain W.

Solution

Let $\theta_t(x_0) = x(t)$. We must have (where defined) $\dot{x}(t)\frac{\partial}{\partial x} = x^2 \frac{\partial}{\partial x}$, so we obtain the differential equation $\dot{x}(t) = x^2(t)$. The solution is $\theta_t(x_0) = \frac{x_0}{1-tx_0}$. We have three cases:

- $x_0 = 0, I_{x_0} = \mathbb{R}$; orbit $\{0\}$;
- $x_0 > 0, I_{x_0} = (-\infty, \frac{1}{x_0})$; orbit $(0, \infty)$;
- $x_0 < 0, I_{x_0} = (\frac{1}{x_0}, \infty)$; orbit $(-\infty, 0)$.

$W = \coprod_{x_0 \in \mathbb{R}} I_{x_0}$.

Remark

We can view $\theta_t(x_0) = x(t)$ as a map from I_{x_0} to the orbit of x_0, a subset of $\mathbb{R}$. Then $\theta_*(x_0)(\frac{\partial}{\partial t}) = x^2 \frac{\partial}{\partial x}$ is an equation between vector fields. Also, if we view $t(x)$ as a function on the orbit of x_0 (the inverse function of $x(t)$, then we can intepret $dt = x^{-2}dx$ as an equation between 1-forms.

4 Aufgabe

Let $M = GL(2, \mathbb{R})$and define an action of $\mathbb{R}$ on M by the formula

$$\theta(t, A) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot A, \qquad A \in GL(2, \mathbb{R}),$$

with the dot denotng matrix multiplication. Find the infinitesimal generator.

Solution Let's choose coordinates $\{x_i\}$ to represent a generic element $A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in GL(2, \mathbb{R})$.

Let $f(x_1, x_2, x_3, x_4)$ denote an element in $C^\infty(A)$, then

$$f \circ \theta_t(x_1, x_2, x_3, x_4) = f(x_1 + tx_3, x_2 + tx_4, x_3, x_4).$$

$$X_A f = \frac{d}{dt} f(x_1 + tx_3, x_2 + tx_4, x_3, x_4)_{|t=0} = x_3 \frac{\partial}{\partial x_1} f_{|A} + x_4 \frac{\partial}{\partial x_2} f_{|A}.$$

Thus $X = x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2}$.