

# Differentialgeometrie II

## Übungsblatt 4 - Lösungen

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### 1 Aufgabe

Let  $M = \mathbb{R}^2$  and  $\theta : \mathbb{R} \times M \rightarrow M$  be given by the formula

$$\theta_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t).$$

- Show that  $\theta$  is a globally defined action of  $\mathbb{R}$  on  $M$ .
- Describe  $X$ , the associated infinitesimal generator.
- Describe the orbits.
- Show that  $X$  is invariant with respect to  $\theta$ , i.e.  $\theta_{t*}(X_{(x,y)}) = X_{\theta_t(x,y)}$ .

*Solution*

- It is pretty evident that  $\theta$  is defined for any  $t \in \mathbb{R}$ , so it is globally defined (by the way, each  $\theta_t$  is a diffeomorphism from  $M$  onto  $M$ ). Matrix multiplication and well known properties of  $\cos(t)$  and  $\sin(t)$  show that  $\theta_t$  gives an action of  $\mathbb{R}$  on  $M$ , i.e.  $\theta_t \circ \theta_s = \theta_{t+s}$  and  $\theta_0 = id$ .
- Let  $p \in M$  and  $f \in C^\infty(p)$ . We write  $f(x, y)$ , where  $x, y$  are usual coordinates for  $\mathbb{R}^2$  and  $p = (x, y)$ . Then  $X_p f = \frac{d}{dt} f \circ \theta_t(x, y)|_{t=0} = \frac{d}{dt} f(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t))|_{t=0} = y \frac{\partial}{\partial x} f(x, y) - x \frac{\partial}{\partial y} f(x, y)$ . Thus  $X_p = y(\frac{\partial}{\partial x})|_p - x(\frac{\partial}{\partial y})|_p$ .
- The flow consists of regular circular motions around the origin (clockwise), thus the orbits are circles..
- Let  $f \in C^\infty(\theta_t(p))$ ,  $p = (x, y)$ .

$$X_{\theta_t(x,y)} f = (-x \sin(t) + y \cos(t)) \frac{\partial}{\partial x} f|_{\theta_t(p)} - (x \cos(t) + y \sin(t)) \frac{\partial}{\partial y} f|_{\theta_t(p)}.$$

$$\begin{aligned} \theta_{t*}(X_{(x,y)}) f &= X_{(x,y)} f \circ \theta_t(x, y) = X_{(x,y)} f(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t)) = \\ &= y \frac{\partial}{\partial x} f(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t))|_p - x \frac{\partial}{\partial y} f(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t))|_p = \\ &= (-x \sin(t) + y \cos(t)) \frac{\partial}{\partial x} f|_{\theta_t(p)} - (x \cos(t) + y \sin(t)) \frac{\partial}{\partial y} f|_{\theta_t(p)}. \end{aligned}$$

### 2 Aufgabe

Let  $M = \mathbb{R}^2$ , the  $x, y$  plane, and  $X = y(\frac{\partial}{\partial x}) + x(\frac{\partial}{\partial y})$ . Find the corresponding domain  $W$  and one-parameter group  $\theta : W \rightarrow M$ .

*Solution* Let  $\theta_t(x_0, y_0) = (x(t), y(t))$ . Then  $\dot{x}(t) = y(t)$  and  $\dot{y}(t) = x(t)$ . The solution is given by  $(x(t), y(t)) = (x_0 \cosh(t) + y_0 \sinh(t), x_0 \sinh(t) + y_0 \cosh(t))$ , which is defined for any  $t \in \mathbb{R}$ . Thus  $W = \mathbb{R} \times M$ .

### 3 Aufgabe

Let  $X = x^2 \frac{\partial}{\partial x}$  be a vector field on  $M = \mathbb{R}$ . Find the associated flow  $\theta$  and describe its domain  $W$ .

*Solution*

Let  $\theta_t(x_0) = x(t)$ . We must have (where defined)  $\dot{x}(t) \frac{\partial}{\partial x} = x^2 \frac{\partial}{\partial x}$ , so we obtain the differential equation  $\dot{x}(t) = x^2(t)$ . The solution is  $\theta_t(x_0) = \frac{x_0}{1-tx_0}$ . We have three cases:

- $x_0 = 0, I_{x_0} = \mathbb{R}$ ; orbit  $\{0\}$ ;
- $x_0 > 0, I_{x_0} = (-\infty, \frac{1}{x_0})$ ; orbit  $(0, \infty)$ ;
- $x_0 < 0, I_{x_0} = (\frac{1}{x_0}, \infty)$ ; orbit  $(-\infty, 0)$ .

$$W = \coprod_{x_0 \in \mathbb{R}} I_{x_0}.$$

*Remark*

We can view  $\theta_t(x_0) = x(t)$  as a map from  $I_{x_0}$  to the orbit of  $x_0$ , a subset of  $\mathbb{R}$ . Then  $\theta_*(x_0)(\frac{\partial}{\partial t}) = x^2 \frac{\partial}{\partial x}$  is an equation between vector fields. Also, if we view  $t(x)$  as a function on the orbit of  $x_0$  (the inverse function of  $x(t)$ ), then we can interpret  $dt = x^{-2} dx$  as an equation between 1-forms.

### 4 Aufgabe

Let  $M = GL(2, \mathbb{R})$  and define an action of  $\mathbb{R}$  on  $M$  by the formula

$$\theta(t, A) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot A, \quad A \in GL(2, \mathbb{R}),$$

with the dot denoting matrix multiplication. Find the infinitesimal generator.

*Solution* Let's choose coordinates  $\{x_i\}$  to represent a generic element  $A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in GL(2, \mathbb{R})$ .

Let  $f(x_1, x_2, x_3, x_4)$  denote an element in  $C^\infty(A)$ , then

$$f \circ \theta_t(x_1, x_2, x_3, x_4) = f(x_1 + tx_3, x_2 + tx_4, x_3, x_4).$$

$$X_A f = \frac{d}{dt} f(x_1 + tx_3, x_2 + tx_4, x_3, x_4)|_{t=0} = x_3 \frac{\partial}{\partial x_1} f|_A + x_4 \frac{\partial}{\partial x_2} f|_A.$$

Thus  $X = x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2}$ .