Topology WS 2006–07

Homework assignment 4, due 22. Nov. 2006

(1) A set $U \subset \mathbb{R}^2$ is called *star-shaped* from $p \in U$ if for every other point $q \in U$ the segment $\overline{pq}$ lies entirely in U. Show that if U is star-shaped (from some point) then $H_1 U = 0$, that is, every closed 1-chain in U is a 1-boundary.

(2) Suppose $U = \mathbb{R}^2 \smallsetminus \{p_1, \dots, p_n\}$. Consider the map $C_1 U \to \mathbb{Z}^n$ given by

$$\gamma \mapsto \big(W(\gamma, p_1), \dots, W(\gamma, p_n)\big).$$

Show this vanishes on boundaries and thus induces a map $\phi : H_1 U \to \mathbb{Z}^n$. Show that this ϕ is an isomorphism.

(3) Suppose U and V are open subsets of $\mathbb{R}^2$ and $F : U \to V$ a continuous map. For a 1-chain $\gamma = \sum n_i \gamma_i$ in U, we define

$$F_* \gamma := \sum n_i F \circ \gamma_i,$$

a 1-chain in V. (And for 0-chains, $F_* \sum n_i p_i := \sum n_i F(p_i)$.) Show that F_* maps 1-cycles to 1-cycles. Show that $F_* \partial \gamma = \partial F_* \gamma$ for all 1-chains γ. Show that F_* maps 1-boundaries to 1-boundaries.