Berlin Mathematical School

Topology WS 2006–07

Homework assignment 1, due 1. Nov. 2006

The first two problems concern open subsets $U \subset \mathbb{R}^2$ in the plane. Remember that U is automatically locally connected: every point in U has a connected neighborhood (for instance a ball in $\mathbb{R}^2$). Thus each connected component of U is simultaneously open and closed.

(1) Prove that a function $f : U \to \mathbb{R}$ is locally constant (that is, each point has a neighborhood on which f is constant) if and only if f is constant on each connected component.

(2) Prove that U is connected if and only if for any two points $P, Q \in U$, there is a (piecewise smooth) path $\gamma : [a, b] \to U$ from P to Q.

(3) Let $p, q \in \mathbb{R}^2$ and let ω_P and ω_Q be the closed 1-forms corresponding to "$d\theta$" around P and Q, respectively. We have seen that $\omega_P - \omega_Q$ is not exact on $\mathbb{R}^2 \smallsetminus \{p, q\}$. Show, however, that it is exact on $\mathbb{R}^2 \smallsetminus \overline{pq}$. What are the level sets of the function f for which $df = \omega_P - \omega_Q$?

(4) Suppose $U \subset \mathbb{R}^2 \smallsetminus \{0\}$ is a subset on which there exists a continuous angle function θ (for which $d\theta = \omega_\theta$). Suppose γ is a closed path in U. Show the winding number $w(\gamma, 0)$ equals 0.