Topology WS 2006–07

Final Exam , due by 12:00 on 16. Feb. 2007

You may use your notes—as well as books or other reference materials—to work on this take-home exam. But you may not discuss these problems with other people. If you have questions, please contact me (for instance by email or at the BMS Days). Please turn in your exam to me (or my secretary, Annett Gillmeister, in MA 320) by noon on Friday.

(1) Let $\mathbb{Z}_n$ denote the group $\mathbb{Z}/n\mathbb{Z}$. Consider singular homology of a space X with integer coefficients and with coefficients in $\mathbb{Z}_n$. Let K_{p-1} be the kernel of the map from $H_{p-1}(X)$ to itself given by multiplication by n. Use the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_n \to 0$$

to derive the exact sequence

$$0 \to H_p(X)/nH_p(X) \to H_p(X;\mathbb{Z}_n) \to K_{p-1} \to 0.$$

(2) Given a space X, its *suspension* ΣX is defined to be the quotient space of $X \times I$ where $X \times \{0\}$ is identified to a point, and so is $X \times \{1\}$. (This is the union of two cones on X.) Use the Eilenberg–Steenrod axioms to show that for any homology theory, there is a natural isomorphism in reduced homology

$$\tilde{H}_p(X) \cong \tilde{H}_{p+1}(\Sigma X).$$

(Hint: our computation of the homology groups of spheres was a special case of this, since $\mathbb{S}^{n+1} = \Sigma\mathbb{S}^n$.)

(3) Suppose X is the figure-eight space, consisting of circles A and B joined at the basepoint x_0. Its fundamental group is

$$\pi_1(X, x_0) = \langle A, B\rangle = \langle A\rangle \star \langle B\rangle \cong \mathbb{Z} \star \mathbb{Z}.$$

(a) Consider the subgroup $H < \pi_1(X)$ generated by the elements A^2 and B^2. What is the covering space of X corresponding to this subgroup H? (Sketch the cover and explain why it corresponds to H.) What is the automorphism group of this cover?

(b) Now consider the map from $\pi_1(X)$ to $\mathbb{Z}_4$ which takes A to 1 and B to -1. The kernel of this map is a normal subgroup $K < \pi_1$, containing for instance A^4 and AB. What is the cover of X corresponding to this subgroup K? (Sketch the cover and explain why it corresponds to K.) What is the automorphism group of this cover?

(4) Let X be the union of the unit sphere in $\mathbb{R}^3$ and the straight line segment connecting its poles (a diameter of the sphere). Use the Seifert/van Kampen theorem to find $\pi_1(X)$.

(5) Suppose X is a simple closed curve in S^2 (that is, a subset of the sphere homeomorphic to a circle). Prove that the complement $S^2 \setminus X$ has exactly two components.

(6) Show that there is no continuous one-to-one map of a Möbius strip into the plane. (Hint: See 5.19, 5.26, 8.14 in Fulton.)