SHEARLET COORBIT SPACES: COMPACTLY SUPPORTED ANALYZING SHEARLETS, TRACES AND EMBEDDINGS

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Abstract. We show that compactly supported functions with sufficient smoothness and enough vanishing moments can serve as analyzing vectors for shearlet coorbit spaces. We use this approach to prove embedding theorems for subspaces of shearlet coorbit spaces resembling shearlets on the cone into Besov spaces. Furthermore, we show embedding relations of traces of these subspaces with respect to the real axes.

1. Introduction

One of the most important tasks in applied analysis is the analysis of signals which are usually modeled by (real or complex valued) functions. The first step is always to decompose the signal with respect to suitable building blocks. There are by now many different ways to choose these building blocks. Most prominent examples are the Fourier, the wavelet and the Gabor transform, respectively. All these different transforms have their advantages and drawbacks, which one to choose depends on the application and on the specific information one wants to extract from the signal. However, for many applications, in particular in image analysis, the wavelet transform is very often the method of choice. Indeed, wavelets are very well suited for piecewise smooth signals with isolated singularities, for in this case the wavelet expansion turns out to be quite sparse which gives rise to very efficient compression strategies. Unfortunately it has been observed that the detection of *directional information* by wavelets is difficult or at least not very efficient. Therefore, in recent years, much effort has been spent to design directional representation systems such as the curvelets [3], the ridgelets [2] and also the shearlets [15] (This list is clearly not complete). Among all these transforms, the shearlet transform stands out since it stems from a square-integrable group representation. This has been clarified in [5] where the underlying group, the *full shearlet group*, has been established. This pure group theoretical approach to shearlets has some important advantages. In particular, it is possible to derive canonical smoothness spaces associated with the shearlet transform. The basic tool to do this is provided by the coorbit space theory derived by Feichtinger and Gröchenig in a series of papers [9, 10, 11]. Under certain additional integrability conditions, the smoothness spaces related with a square-integrable group representation are defined by the decay of the associated voice transform. This technique is quite universal, and the classical smoothness spaces such as Besov and modulation spaces can be interpreted as coorbit spaces associated with the affine group and the Weyl-Heisenberg group, respectively. Moreover, the coorbit space theory provides a very general discretization technique which produces atomic decompositions and Banach frames for the coorbit spaces.

In [6], it has been clarified that the coorbit theory is indeed applicable to the full shearlet group. Moreover, in [7], a natural generalization to arbitrary space dimensions has been derived.

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However, once these new smoothness spaces, the shearlet coorbit spaces, are established, some natural questions arise. How do these spaces really look like? Are there 'nice' sets of functions that are dense in these spaces? What are the relations to classical smoothness spaces such as Besov spaces? Do there exist embeddings into Besov spaces? And do there exist generalized versions of Sobolev embedding theorems for shearlet coorbit spaces? Moreover, can the associated trace spaces be identified? In this paper, we provide some first answers to these questions. We concentrate on the two-dimensional case, and we show that

- For large classes of weights, variants of Sobolev embeddings exist;
- for natural subclasses which in a certain sense correspond to the 'cone adapted shearlets' [18], there exist embeddings into (homogeneous) Besov spaces:
- for the same subclass, the traces onto the coordinate axis can again be identified with homogeneous Besov spaces.

Our approach heavily relies on atomic decomposition techniques. Recall that the coorbit space theory naturally gives rise to Banach frames, and therefore, by using the associated norm equivalences, all the tasks outlined above can be studied by means of weighted sequence. In particular, based on the general analysis in [16], quite recently this technique has been applied to derive new embedding and trace results for Besov spaces [20]. The analysis presented in this paper was partially inspired by this thesis.

To make this approach really powerful, it is very convenient and sometimes even necessary to work with compactly supported building blocks. In the shearlet case, this is a nontrivial problem, since usually the analyzing shearlets are band-limited functions. For the specific case of shearlets on the cone, quite recently a first solution has been provided in [17]. We refer to the overview article [19] for a detailed discussion. Since the shearlets on the cone do not really fit into the group theoretical setting, we provide a new construction of families of compactly supported shearlets in this paper. We show that indeed a compactly supported function with sufficient smoothness and enough vanishing moments can serve as an analyzing vector for shearlet coorbit spaces.

This paper is organized as follows: We start by introducing the shearlet group in Section 2. In Section 3 we consider shearlet coorbit spaces, their atomic decompositions and shearlet Banach frames. We show that compactly supported shearlets can be used as analyzing vectors for these spaces, in particular there exist compactly supported shearlets within so-called 'better' sets used to define atomic decompositions. Finally we prove that the Schwartz functions are dense in our shearlet coorbit spaces. Section 4 deals with relations between shearlet coorbit spaces and Besov spaces. After recalling the general characteristics of homogeneous Besov spaces we prove embeddings of traces of certain subspaces of shearlet coorbit spaces on the real axes into (sums of) onedimensional Besov spaces. Finally, we show that these shearlet coorbit subspaces are themselves embedded into (sums of) Besov spaces of appropriate smoothness.

2. The Shearlet Group

In this section, we provide the basic notation and results about the shearlet group and its square integrable representations including the corresponding admissible functions, the so-called shearlets. For $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}$, let

$$A_a := \begin{pmatrix} a & 0 \\ 0 & \operatorname{sgn}(a)\sqrt{|a|} \end{pmatrix} \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

denote the *parabolic scaling matrix* and the *shear matrix*, respectively, where $\operatorname{sgn}(a)$ denotes the sign of a. The *(full) shearlet group* $\mathbb{S}$ is defined to be the set $\mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^2$ endowed with the group operation

$$(a,s,t)\,(a',s',t') = (aa', s + s'\sqrt{|a|}, t + S_s A_a t').$$

A left–invariant and right-invariant Haar measures of $\mathbb{S}$ is given by

$$\mu_{\mathbb{S},l} = \frac{da}{|a|^3}\,ds\,dt \quad \text{and} \quad \mu_{\mathbb{S},r} = \frac{da}{|a|}\,ds\,dt,$$

respectively and the modular function of $\mathbb{S}$ by $\triangle(a,s,t) = 1/|a|^2$. In the following, we use the left-invariant Haar measure $\mu_{\mathbb{S}} = \mu_{\mathbb{S},l}$. Let L_x, R_x denote the left and right translations by $x \in \mathbb{S}$, i.e., $L_x F(y) := F(x^{-1}y)$ and $R_x F(y) := F(yx)$.

Recall that a *unitary representation* of a locally compact group G with the left–invariant Haar measure μ_G on a Hilbert space $\mathcal{H}$ is a homomorphism π from G into the group of unitary operators $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$ which is continuous with respect to the strong operator topology. For the shearlet group the mapping $\pi : \mathbb{S} \to \mathcal{U}(L_2(\mathbb{R}^2))$ defined by

$$\pi(a,s,t)\,\psi(x) := |a|^{-\frac{3}{4}}\,\psi(A_a^{-1}S_s^{-1}(x-t)) = |a|^{-\frac{3}{4}}\,\psi\left(\frac{1}{a}\,(x_1 - t_1 - s(x_2 - t_2))\,, \frac{\operatorname{sgn} a}{\sqrt{|a|}}\,(x_2 - t_2)\right) \tag{1}$$

is a unitary representation of $\mathbb{S}$, see [5, 6]. In the following, we use the abbreviation $\psi_{a,s,t} := \pi(a,s,t)\,\psi$. Let the Fourier transform be defined by

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i \langle \omega, x\rangle}\,dx.$$

Then straightforward computation yields

$$\hat{\psi}_{a,s,t}(\omega) = |a|^{\frac{3}{4}} e^{-2\pi i t\omega}\hat{\psi}\left(A_a^{\mathrm{T}} S_s^{\mathrm{T}}\omega\right) = |a|^{\frac{3}{4}} e^{-2\pi i t\omega}\hat{\psi}\left(a\omega_1, \operatorname{sgn}(a)\sqrt{|a|}(s\omega_1 + \omega_2)\right). \tag{2}$$

A function $\psi \in L_2(\mathbb{R}^2)$ is called *admissible*, if

$$\int_{\mathbb{S}} |\langle \psi, \pi(g)\psi\rangle|^2 d\mu_{\mathbb{S}}(g) < \infty.$$

If a unitary representation π is irreducible and there exists at least one admissible function $\psi \in L_2(\mathbb{R}^2)$ then π is called *square integrable*.

The following result from [6] shows that the unitary representation π defined in (1) is a square-integrable representation of $\mathbb{S}$.

Theorem 2.1. *A function $\psi \in L_2(\mathbb{R}^2)$ is admissible if and only if it fulfills the* admissibility condition

$$C_\psi := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\hat{\psi}(\omega_1,\omega_2)|^2}{\omega_1^2}\,d\omega_2\,d\omega_1 < \infty. \tag{3}$$

Then, for any $f \in L^2(\mathbb{R}^2)$, the following equality holds true:

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2\,d\mu_{\mathbb{S}}(a,s,t) = C_\psi\,\|f\|_2^2\,. \tag{4}$$

In particular, the unitary representation π is irreducible and hence square-integrable.

A function $\psi \in L_2(\mathbb{R}^2)$ fulfilling the admissibility condition (3) is called a *continuous shearlet*. The transform $\mathcal{SH}_\psi : L_2(\mathbb{R}^2) \to L_2(\mathbb{S})$ defined by

$$\mathcal{SH}_\psi f(a,s,t) = \langle f, \psi_{a,s,t}\rangle \tag{5}$$

is called *Continuous Shearlet Transform.* The admissibility condition is important, since it implies a resolution of identity that allows the reconstruction of a function $f \in L_2(\mathbb{R}^2)$ from its Continuous Shearlet Transform.

3. Shearlet Coorbit Spaces from Shearlets with Compact Support

3.1. **Shearlet Coorbit Spaces.** Let w be real-valued, continuous, submultiplicative weight on $\mathbb{S}$, i.e., $w(gh) \le w(g)w(h)$ for all $g, h \in \mathbb{S}$. Furthermore, we will always assume that the weight function w satisfies all the coorbit-theory conditions as stated in [14, Section 2.2].

To define our coorbit spaces we need the set

$$\mathcal{A}_w := \{\psi \in L_2(\mathbb{R}^2) : \mathcal{SH}_\psi(\psi) = \langle \psi, \pi(\cdot)\psi \rangle \in L_{1,w}\}.$$

of *analyzing vectors.* In particular, we assume that our weight is symmetric with respect to the modular function, i.e., $w(g) = w(g^{-1})\triangle(g^{-1})$. Starting with an ordinary weight function w, its symmetric version can be obtained by $w^{\#}(g) := w(g) + w(g^{-1})\triangle(g^{-1})$. Moreover, it was proved in Lemma 2.4 of [9] that $\mathcal{A}_w = \mathcal{A}_{w^{\#}}$.

We want to show that $\mathcal{A}_w$ contains shearlets with compact support. To this end, we need the following auxiliary lemma which is a modification of Lemma 11.1.1 in [12].

Lemma 3.1. *For $r > 1$ and $\alpha > 0$, the following estimate holds true*

$$I(x) := \int_{\mathbb{R}} (1+|t|)^{-r} (1+\alpha|x-t|)^{-r} \, dt \le C \left(\frac{1}{\alpha}(1+|x|)^{-r} + (1+\alpha|x|)^{-r} \right).$$

Note that for $\alpha = 1$ the above property is known as weak subconvolutivity which holds true for more general weight functions, see [8].

Proof. Let

$$\mathcal{N}_x := \left\{ t \in \mathbb{R} : |t-x| \le \frac{|x|}{2} \right\}, \quad \mathcal{N}_x^c := \left\{ t \in \mathbb{R} : |t-x| > \frac{|x|}{2} \right\}.$$

Then we obtain for $t \in \mathcal{N}_x$ by $|x| - |t| \le |t-x| \le |x|/2$ that $|t| \ge |x|/2$ and consequently

$$(1+|t|)^{-r} \le \left(1 + \frac{|x|}{2}\right)^{-r} \le 2^r (1+|x|)^{-r}.$$

Now the above integral can be estimated as follows:

$$\begin{aligned} I(x) &= \int_{\mathcal{N}_x} (1+|t|)^{-r}(1+\alpha|x-t|)^{-r} \, dt + \int_{\mathcal{N}_x^c} (1+|t|)^{-r}(1+\alpha|x-t|)^{-r} \, dt \\ &\le 2^r(1+|x|)^{-r} \int_{\mathcal{N}_x} (1+\alpha|x-t|)^{-r} \, dt + \left(1 + \alpha \frac{|x|}{2}\right)^{-r} \int_{\mathcal{N}_x^c} (1+|t|)^{-r} \, dt \\ &\le 2^r \frac{1}{\alpha}(1+|x|)^{-r} \int_{\mathbb{R}} (1+|u|)^{-r} \, du + 2^r(1+\alpha|x|)^{-r} \int_{\mathbb{R}} (1+|t|)^{-r} \, dt. \end{aligned}$$

This implies the assertion. □

For some $D > 0$, let $Q_D := [-D, D] \times [-D, D]$. The following theorem shows that $\mathcal{A}_w$ contains shearlets with compact support.

Theorem 3.2. *For some $D > 0$, let $\psi(x) \in L_2(\mathbb{R}^2)$ fulfill $\operatorname{supp} \psi \in Q_D$. Suppose that the weight function satisfies $w(a, s, t) = w(a) \le |a|^{-\rho_1} + |a|^{\rho_2}$ for $\rho_1, \rho_2 \ge 0$ and that*

$$|\hat{\psi}(\omega_1, \omega_2)| \le C \frac{|\omega_1|^n}{(1+|\omega_1|)^r} \frac{1}{(1+|\omega_2|)^r} \qquad (6)$$

with $n \geq \max(\frac{1}{4}+\rho_2, \frac{9}{4}+\rho_1)$ *and* $r > n+\max(\frac{7}{4}+\rho_2, \frac{9}{4}+\rho_1)$. *Then we have that* $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$, *i.e.,*

$$I := \int_{\mathbb{S}} |\mathcal{SH}_\psi(\psi)(g)|\, w(g)\, d\mu(g) < \infty.$$

Proof. First we have by the support property of ψ that $\mathcal{SH}_\psi(\psi) = \langle \psi, \psi_{a,s,t}\rangle \neq 0$ requires $(x_1, x_2) \in Q_D$ and

$$\begin{aligned} -D &\leq \frac{\operatorname{sgn} a}{\sqrt{|a|}}\,(x_2 - t_2) \leq D, \\ -D &\leq \frac{1}{a}\,(x_1 - t_1 - s(x_2 - t_2)) \leq D. \end{aligned}$$

Hence $\langle \psi, \psi_{a,s,t}\rangle \neq 0$ implies that

$$\begin{aligned} -D(1+\sqrt{|a|}) &\leq t_2 \leq D(1+\sqrt{|a|}), \\ -D\left(1+|a|+|s|(2+\sqrt{|a|})\right) &\leq t_1 \leq D\left(1+|a|+|s|(2+\sqrt{|a|})\right). \end{aligned}$$

Using this relation we obtain that

$$I \leq \int_{\mathbb{R}^*}\int_{\mathbb{R}} 4D^2(1+\sqrt{|a|})\left(1+|a|+|s|(2+\sqrt{|a|})\right)|\langle\psi,\psi_{a,s,t}\rangle|\, ds\, w(a)\frac{da}{|a|^3}.$$

Next, Plancherel's equality together with (2) and the decay assumptions on $\hat{\psi}$ yield

$$\begin{aligned} I &\leq C\int_{\mathbb{R}^*}\int_{\mathbb{R}}(1+\sqrt{|a|})\left(1+|a|+|s|(2+\sqrt{|a|})\right)|\langle\hat{\psi},\hat{\psi}_{a,s,t}\rangle|\, ds\, w(a)\frac{da}{|a|^3} \\ &\leq C\int_{\mathbb{R}^*}\int_{\mathbb{R}}\left(\underbrace{1+|a|^{\frac{1}{2}}+|a|+|a|^{\frac{3}{2}}}_{=:p_3(|a|^{\frac{1}{2}})}+|s|(\underbrace{2+3|a|^{\frac{1}{2}}+a}_{=:p_2(|a|^{\frac{1}{2}})})\right)J(a,s)\, ds\, w(a)\frac{da}{|a|^3} \end{aligned}$$

where $p_k \in \Pi_k$ are polynomials of degree $\leq k$, $|\mathcal{SH}_\psi\psi(a,s,t)| \leq J(a,s)$ and

$$\begin{aligned} J(a,s) &:= |a|^{\frac{3}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|\omega_1|^n}{(1+|\omega_1|)^r}\frac{1}{(1+|\omega_2|)^r}\frac{|a\omega_1|^n}{(1+|a\omega_1|)^r}\frac{1}{(1+\sqrt{|a|}\,|s\omega_1+\omega_2|)^r}\, d\omega_2 d\omega_1 \\ &= \int_{\mathbb{R}}\frac{|\omega_1|^n}{(1+|\omega_1|)^r}\frac{|a\omega_1|^n}{(1+|a\omega_1|)^r}\int_{\mathbb{R}}\frac{1}{(1+|\omega_2|)^r}\frac{1}{(1+\sqrt{|a|}\,|s\omega_1+\omega_2|)^r}\, d\omega_2 d\omega_1. \end{aligned}$$

The inner integral can be estimated by Lemma 3.1 which results in

$$J(a,s) \leq C\,|a|^{n+\frac{3}{4}}\int_{\mathbb{R}}\frac{|\omega_1|^n}{(1+|\omega_1|)^r}\frac{|\omega_1|^n}{(1+|a\omega_1|)^r}\left(\frac{1}{\sqrt{|a|}\,(1+|s\omega_1|)^r}+\frac{1}{(1+\sqrt{|a|}\,|s\omega_1|)^r}\right)d\omega_1. \quad (7)$$

Now we obtain

$$\begin{aligned} I \leq\; & C\Bigg(\int_{\mathbb{R}^*}\int_{\mathbb{R}}\int_{\mathbb{R}}|a|^{n-\frac{11}{4}}\,(p_3+|s|p_2)\,\frac{|\omega_1|^{2n}}{(1+|\omega_1|)^r(1+|a\omega_1|)^r}\frac{1}{(1+|s\omega_1|)^r}\, ds d\omega_1 w(a) da \\ &+\int_{\mathbb{R}^*}\int_{\mathbb{R}}\int_{\mathbb{R}}|a|^{n-\frac{9}{4}}\,(p_3+|s|p_2)\,\frac{|\omega_1|^{2n}}{(1+|\omega_1|)^r(1+|a\omega_1|)^r}\frac{1}{(1+\sqrt{|a|}\,|s\omega_1|)^r}\, ds d\omega_1 w(a) da\Bigg). \end{aligned}$$

Since the integrand is even in ω_1, s and a this can be further simplified as

$$\begin{aligned} I &\leq C\Bigg(\int_0^\infty a^{n-\frac{11}{4}} p_3(\sqrt{a}) \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+a\omega_1)^r} \int_0^\infty \frac{1}{(1+s\omega_1)^r}\, ds d\omega_1 w(a) da \\ &\quad + \int_0^\infty a^{n-\frac{11}{4}} p_2(\sqrt{a}) \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+a\omega_1)^r} \int_0^\infty \frac{s}{(1+s\omega_1)^r}\, ds d\omega_1 w(a) da \\ &\quad + \int_0^\infty a^{n-\frac{9}{4}} p_3(\sqrt{a}) \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+a\omega_1)^r} \int_0^\infty \frac{1}{(1+\sqrt{a}s\omega_1)^r}\, ds d\omega_1 w(a) da \\ &\quad + \int_0^\infty a^{n-\frac{9}{4}} p_2(\sqrt{a}) \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+a\omega_1)^r} \int_0^\infty \frac{s}{(1+\sqrt{a}s\omega_1)^r}\, ds d\omega_1 w(a) da\Bigg). \end{aligned}$$

Substituting $t := s\omega_1$ with $dt = \omega_1\, ds$ in the first two integrals and $t := \sqrt{a}s\omega_1$ with $dt = \sqrt{a}\omega_1\, ds$ in the last two integrals, we obtain for $r > 2$ that

$$\begin{aligned} I &\leq C\Bigg(\int_0^\infty \frac{\omega_1^{2n-1}}{(1+\omega_1)^r} \int_0^\infty a^{n-\frac{11}{4}} p_3(\sqrt{a}) \frac{1}{(1+a\omega_1)^r}\, w(a) da d\omega_1 \\ &\quad + \int_0^\infty \frac{\omega_1^{2n-2}}{(1+\omega_1)^r} \int_0^\infty a^{n-\frac{11}{4}} p_2(\sqrt{a}) \frac{1}{(1+a\omega_1)^r}\, w(a) da d\omega_1 \\ &\quad + \int_0^\infty \frac{\omega_1^{2n-1}}{(1+\omega_1)^r} \int_0^\infty a^{n-\frac{11}{4}} p_3(\sqrt{a}) \frac{1}{(1+a\omega_1)^r}\, w(a) da d\omega_1 \\ &\quad + \int_0^\infty \frac{\omega_1^{2n-2}}{(1+\omega_1)^r} \int_0^\infty a^{n-\frac{13}{4}} p_2(\sqrt{a}) \frac{1}{(1+a\omega_1)^r}\, w(a) da d\omega_1\Bigg). \end{aligned}$$

Substituting $b := a\omega_1$ with $db = \omega_1 da$ and bounding w accordingly we conclude further that

$$\begin{aligned} I &\leq C\Bigg(\int_0^\infty \frac{\omega_1^{n+\frac{3}{4}+\rho_1}}{(1+\omega_1)^r} \int_0^\infty p_3\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{11}{4}-\rho_1}}{(1+b)^r}\, db d\omega_1 \\ &\quad + \int_0^\infty \frac{\omega_1^{n-\frac{1}{4}+\rho_1}}{(1+\omega_1)^r} \int_0^\infty p_2\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{11}{4}-\rho_1}}{(1+b)^r}\, db d\omega_1 \\ &\quad + \int_0^\infty \frac{\omega_1^{n+\frac{1}{4}+\rho_1}}{(1+\omega_1)^r} \int_0^\infty p_2\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{13}{4}-\rho_1}}{(1+b)^r}\, db d\omega_1 \\ &\quad + \int_0^\infty \frac{\omega_1^{n+\frac{3}{4}-\rho_2}}{(1+\omega_1)^r} \int_0^\infty p_3\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{11}{4}+\rho_2}}{(1+b)^r}\, db d\omega_1 \\ &\quad + \int_0^\infty \frac{\omega_1^{n-\frac{1}{4}-\rho_2}}{(1+\omega_1)^r} \int_0^\infty p_2\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{11}{4}+\rho_2}}{(1+b)^r}\, db d\omega_1 \\ &\quad + \int_0^\infty \frac{\omega_1^{n+\frac{1}{4}-\rho_2}}{(1+\omega_1)^r} \int_0^\infty p_2\left(\sqrt{\frac{b}{\omega_1}}\right) \frac{b^{n-\frac{13}{4}+\rho_2}}{(1+b)^r}\, db d\omega_1\Bigg). \end{aligned}$$

Since $p_k \in \Pi_k$, $k = 2, 3$ we see that the integrals are finite if $n \geq \max(\frac{1}{4} + \rho_2, \frac{9}{4} + \rho_1)$ and $r > n + \max(\frac{7}{4} + \rho_2, \frac{9}{4} + \rho_1)$. This finishes the proof. □

For an analyzing vector ψ we can consider the space

$$\mathcal{H}_{1,w} := \{f \in L_2(\mathbb{R}^2) : \mathcal{SH}_\psi(f) = \langle f, \pi(\cdot)\psi\rangle \in L_{1,w}(\mathbb{S})\}, \tag{8}$$

with norm $\|f\|_{\mathcal{H}_{1,w}} := \|\mathcal{SH}_\psi f\|_{L_{1,w}(\mathbb{S})}$ and its anti-dual $\mathcal{H}^\sim_{1,w}$, the space of all continuous conjugate-linear functionals on $\mathcal{H}_{1,w}$. The spaces $\mathcal{H}_{1,w}$ and $\mathcal{H}^\sim_{1,w}$ are π-invariant Banach spaces with continuous embedding $\mathcal{H}_{1,w} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^\sim_{1,w}$. Then the inner product on $L_2(\mathbb{R}^2)\times L_2(\mathbb{R}^2)$ extends to a sesquilinear form on $\mathcal{H}^\sim_{1,w} \times \mathcal{H}_{1,w}$, therefore for $\psi \in \mathcal{H}_{1,w}$ and $f \in \mathcal{H}^\sim_{1,w}$ the *extended representation coefficients*

$$\mathcal{SH}_\psi(f)(a,s,t) := \langle f, \pi(a,s,t)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}}$$

are well-defined.

Let m be a w-moderate weight on $\mathbb{S}$ which means that $m(xyz) \le w(x)m(y)w(z)$ for all $x, y, z \in \mathbb{S}$. For $1 \le p \le \infty$, let

$$L_{p,m}(\mathbb{S}) := \{F \text{ measurable} : Fm \in L_p(\mathbb{S})\}.$$

We are interested in the following Banach spaces which are called *shearlet coorbit spaces*

$$\mathcal{SC}_{p,m} := \{f \in \mathcal{H}^\sim_{1,w} : \mathcal{SH}_\psi(f) \in L_{p,m}(\mathbb{S})\}, \quad \|f\|_{\mathcal{SC}_{p,m}} := \|\mathcal{SH}_\psi f\|_{L_{p,m}(\mathbb{S})}. \tag{9}$$

Note that the definition of $\mathcal{SC}_{p,m}$ is independent of the analyzing vector ψ and of the weight w in the sense that $\tilde{w}$ with $w(g) \le C\tilde{w}(g)$ for all $g \in \mathbb{S}$ and $\mathcal{A}_{\tilde{w}} \ne \{0\}$ give rise to the same space see [9, Theorem 4.2].

In applications, one may start with some sub-multiplicative weight m and use the symmetric weight $w := m^{\#}$ for the definition of $\mathcal{A}_w$. Obviously, we have that m is w-moderate.

3.2. **Atomic Decompositions and Shearlet Banach Frames.** To construct atomic decompositions and Banach frames of our shearlet coorbit spaces the following better subset $\mathcal{B}_w$ of $\mathcal{A}_w$ has to be non-empty:

$$\mathcal{B}_w := \{\psi \in L_2(\mathbb{R}^2) : \mathcal{SH}_\psi(\psi) \in \mathcal{W}(C_0, L_{1,w})\},$$

where $\mathcal{W}(C_0, L_{1,w})$ is the Wiener-Amalgam space

$$\mathcal{W}(C_0, L_{1,w}) := \{F : \|(L_x\chi_{\mathcal{Q}})F\|_\infty \in L_{1,w}\}, \quad \|(L_x\chi_{\mathcal{Q}})F\|_\infty = \sup_{y\in x\mathcal{Q}} |F(y)|$$

and $\mathcal{Q}$ is a relatively compact neighborhood of the identity element in $\mathbb{S}$, see [14]. Note that in general $\mathcal{B}_w$ is defined with respect to the right version $\mathcal{W}^R(C_0, L_{1,w}) := \{F : \|(R_x\chi_Q)F\|_\infty = \sup_{y\in\mathcal{Q}x^{-1}} |F(y)| \in L_{1,w}\}$ of the Wiener-Amalgam space. Regarding that $\mathcal{SH}_\psi(\psi)(g) = \mathcal{SH}_\psi\psi(g^{-1})$ and assuming that $\mathcal{Q} = \mathcal{Q}^{-1}$ both definitions of $\mathcal{B}_w$ coincide.

Corollary 3.3. *Let $\psi(x) \in L_2(\mathbb{R}^2)$ fulfill $\operatorname{supp}\psi \in Q_D$. Suppose that the weight function satisfies $w(a,s,t) = w(a) \le |a|^{-\rho_1} + |a|^{\rho_2}$ for $\rho_1, \rho_2 \ge 0$ and that*

$$|\hat{\psi}(\omega_1,\omega_2)| \le C\frac{|\omega_1|^n}{(1+|\omega_1|)^r}\,\frac{1}{(1+|\omega_2|)^r} \tag{10}$$

for sufficiently large n and r. Then we have that $\psi \in \mathcal{B}_w$.

Proof. To keep technicalities at a reasonable level, we restrict ourselves to the case $w \equiv 1$. Let $\mathcal{Q} = \mathcal{Q}^{-1} \subset [\frac{1}{\alpha}, \alpha] \times [-\sigma, \sigma] \times Q_\tau$, where $\alpha > 1$, $\sigma, \tau > 0$. In the following, we restrict our attention to group elements of $\mathbb{S}$ with $a > 0$ and $s \ge 0$. The other cases can be deduced in a similar way. Let $(a_q, s_q, t_q) \in \mathcal{Q}$ and

$$(a', s', t') := (a,s,t)(a_q,s_q,t_q) = \left(aa_q, s + s_q\sqrt{a}, \begin{pmatrix} t_1 + at_{q,1} + \sqrt{a}\,s\,t_{q,2} \\ t_2 + \sqrt{a}\,t_{q,2}\end{pmatrix}\right).$$

We are interested in

$$G(a,s,t) := \sup_{(a_q,s_q,t_q)\in\mathcal{Q}} |\mathcal{SH}_\psi\psi(a',s',t')|.$$

As in the proof of Theorem 3.2 we have that $\mathcal{SH}_\psi\psi(a',s',t')$ is zero if t' does not fulfill

$$\begin{aligned} -D &\le \frac{1}{\sqrt{a'}}\,(x_2 - t_2') \le D, \\ -D &\le \frac{1}{a'}\left(x_1 - t_1' - s'(x_2 - t_2')\right) \le D, \end{aligned}$$

where $x \in Q_D$. By definition of a', s', t' this implies that

$$\begin{aligned} x_2 - \sqrt{a}\,t_{q,2} - D\sqrt{a_q a} &\le t_2 \le x_2 - \sqrt{a}\,t_{q,2} + D\sqrt{a_q a}, \\ r - Daa_q &\le t_1 \le r + Daa_q, \end{aligned}$$

where $r := x_1 - at_{q,1} - \sqrt{a}\,s\,t_{q,2} - (s + s_q\sqrt{a})(x_2 - t_2 - \sqrt{a}\,t_{q,2})$. By definition of $\mathcal{Q}$ we conclude that $G(a,s,t)$ becomes zero if t is not contained in

$$\begin{aligned} -C(1+\sqrt{a}) &\le t_2 \le C(1+\sqrt{a}), \qquad C := \max\{D, D(\sqrt{\alpha}+\tau\}, \\ -P_2(\sqrt{a}) - s\,P_1(\sqrt{a}) &\le t_1 \le P_2(\sqrt{a}) + s\,P_1(\sqrt{a}), \end{aligned}$$

where $P_k \in \Pi_k$ are polynomials with nonnegative coefficients depending on α, σ and τ. As in the proof of Theorem 3.2 we conclude that $|\mathcal{SH}_\psi\psi(a',s',t')| \le C\,J(a',s')$, where

$$\begin{aligned} J(a',s') &:= (a')^{n+\frac{3}{4}} \int_{\mathbb{R}} \frac{|\omega_1|^{2n}}{(1+|\omega_1|)^r(1+|a'\omega_1|)^r} \left(\frac{1}{\sqrt{a'}\,(1+|s'\omega_1|)^r} + \frac{1}{(1+\sqrt{a'}\,|s'\omega_1|)^r} \right) d\omega_1 \\ &= (aa_q)^{n+\frac{3}{4}} \int_{\mathbb{R}} \frac{|\omega_1|^{2n}}{(1+|\omega_1|)^r(1+|aa_q\omega_1|)^r} \left(\frac{1}{\sqrt{aa_q}\,(1+|s+s_q\sqrt{a}||\omega_1|)^r} \right) d\omega_1 \\ &\quad + (aa_q)^{n+\frac{3}{4}} \int_{\mathbb{R}} \frac{|\omega_1|^{2n}}{(1+|\omega_1|)^r(1+|aa_q\omega_1|)^r} \left(\frac{1}{(1+\sqrt{aa_q}\,|s+s_q\sqrt{a}||\omega_1|)^r} \right) d\omega_1. \end{aligned}$$

For $0 \le s \le 2\sigma\sqrt{a}$ we use the estimate $|s + s_q\sqrt{a}| \ge 0$ to get

$$|G(a,s,t)| \le Ca^{n+\frac{3}{4}} \left(\frac{1}{\sqrt{a}} + 1 \right) \int_{\mathbb{R}} \frac{|\omega_1|^{2n}}{(1+|\omega_1|)^r(\frac{1}{\alpha}+|a\omega_1|)^r}\,d\omega_1.$$

For $s > 2\sigma\sqrt{a}$ we have that $|s + s_q\sqrt{a}| \ge |s|/2$ and consequently

$$|G(a,s,t)| \le Ca^{n+\frac{3}{4}} \int_{\mathbb{R}} \frac{|\omega_1|^{2n}}{(1+|\omega_1|)^r(\frac{1}{\alpha}+|a\omega_1|)^r} \left(\frac{1}{\sqrt{a}(2+|s||\omega_1|)^r} + \frac{1}{(\frac{2}{\alpha}+\sqrt{a}|s||\omega_1|)^r} \right) d\omega_1.$$

If the following integral is finite, then we can conclude that $G \in L_{1,w}$:

$$\begin{aligned} I &:= \int_{\mathbb{S}} |G(a,s,t)|\,dt\,ds\,\frac{da}{|a|^3} \\ &\le C\int_0^\infty \int_0^{2\sigma\sqrt{a}} \left(p_3(\sqrt{a}) + sp_2(\sqrt{a})\right)|G(a,s,t)|\,ds\,\frac{da}{a^3} \\ &\quad + C\int_0^\infty \int_{2\sigma\sqrt{a}}^\infty \left(p_3(\sqrt{a}) + sp_2(\sqrt{a})\right)|G(a,s,t)|\,ds\,\frac{da}{a^3}, \end{aligned}$$

where $p_3(\sqrt{a}) := (1+\sqrt{a})P_2(\sqrt{a}) \in \Pi_3$ and $p_2(\sqrt{a}) := (1+\sqrt{a})P_1(\sqrt{a}) \in \Pi_2$. By the above estimates of G this can be rewritten as

$$\begin{aligned} I &\le C\int_0^\infty\int_0^\infty a^{n-\frac{9}{4}}\left(p_3(\sqrt{a})+\tilde{p}_3(\sqrt{a})\right)(\sqrt{a}+1)\frac{\omega_1^{2n}}{(1+\omega_1)^r(\frac{1}{\alpha}+a\omega_1)^r}\,da\,d\omega_1 \\ &+ C\int_0^\infty\int_0^\infty\int_0^\infty a^{n-\frac{11}{4}}\left(p_3(\sqrt{a})+sp_2(\sqrt{a})\right)\frac{\omega_1^{2n}}{(1+\omega_1)^r(\frac{1}{\alpha}+a\omega_1)^r}\frac{1}{(2+s\omega_1)^r}\,dads d\omega_1 \\ &+ C\int_0^\infty\int_0^\infty\int_0^\infty a^{n-\frac{9}{4}}\left(p_3(\sqrt{a})+sp_2(\sqrt{a})\right)\frac{\omega_1^{2n}}{(1+\omega_1)^r(\frac{1}{\alpha}+a\omega_1)^r}\frac{1}{(\frac{2}{\alpha}+\sqrt{a}s\omega_1)^r}\,dads d\omega_1, \end{aligned}$$

where $\tilde{p}_3(\sqrt{a}) := 2\sigma\sqrt{a}p_2(\sqrt{a}) \in \Pi_3$. The two later integrals can be estimated as in the proof of Theorem 3.2 with $\rho_1 = \rho_2 = 0$ and the first integral by

$$I_1 \le C\int_0^\infty \frac{\omega_1^{n+\frac{5}{4}}}{(1+\omega_1)^r}\int_0^\infty p_4\left(\sqrt{\frac{b}{\omega_1}}\right)\frac{b^{n-\frac{9}{4}}}{(\frac{1}{\alpha}+b)^r}\,dbd\omega_1,$$

where $p_4(\sqrt{a}) = \left(p_3(\sqrt{a})+\tilde{p}_3(\sqrt{a})\right)(\sqrt{a}+1)$. This integral is finite for $n \ge \frac{9}{4}$ and $r > n+\frac{9}{4}$. This completes the proof. □

A (countable) family $X = \{(a_i,s_i,t_i) : i \in \mathcal{I}\}$ in $\mathbb{S}$ is said to be *U-dense* if $\cup_{i\in\mathcal{I}}(a_i,s_i,t_i)U = \mathbb{S}$, and *separated* if for some compact neighborhood Q of e we have $(a_i,s_i,t_i)Q \cap (a_j,s_j,t_j)Q = \emptyset, i \ne j$, and *relatively separated* if X is a finite union of separated sets. Let $\alpha > 1$ and $\sigma,\tau > 0$ be defined such that

$$[\frac{1}{\alpha},\alpha) \times [-\sigma,\sigma) \times Q_\tau \subset U. \tag{11}$$

Then it was shown in [6] that the set

$$X := \{(\epsilon\alpha^{-j}, \sigma\alpha^{-j/2}k, S_{\sigma\alpha^{-j/2}k}A_{\alpha^{-j}}\tau l) : j \in \mathbb{Z}, k \in \mathbb{Z}, l \in \mathbb{Z}^2, \epsilon \in \{-1,1\}\} \tag{12}$$

is U-dense and relatively separated. For $g \in \mathbb{S}$, let

$$\operatorname{osc}_{\psi,U}(g) := \sup_{u\in U}|\mathcal{SH}_\psi\psi(ug) - \mathcal{SH}_\psi\psi(g)|.$$

The following theorem from [6] shows that the functions in our shearlet coorbit spaces possess atomic decompositions.

Theorem 3.4. *Let $p \in [1,\infty]$. Assume that the irreducible, unitary representation π is w-integrable and let $\psi \in \mathcal{B}_w$. Choose a neighborhood U of e so small that*

$$\|\operatorname{osc}_{\psi,U}\|_{L_{1,w}(\mathbb{S})} < 1. \tag{13}$$

Then for any U-dense and relatively separated set $X = \{(a_i,s_i,t_i) : i \in \mathcal{I}\}$ the space $\mathcal{SC}_{p,m}$ has the following atomic decomposition*: If $f \in \mathcal{SC}_{p,m}$, then*

$$f = \sum_{i\in\mathcal{I}} c_i(f)\pi(a_i,s_i,t_i)\psi, \tag{14}$$

where the sequence of coefficients depends linearly on f and satisfies

$$\|(c_i(f))_{i\in\mathcal{I}}\|_{\ell_{p,m}} \le C\|f\|_{\mathcal{SC}_{p,m}} \tag{15}$$

with a constant C depending only on ψ and with $\ell_{p,m}$ being defined by

$$\ell_{p,m} := \{c = (c_i)_{i\in\mathcal{I}} : \|c\|_{\ell_{p,m}} := \|c\,m\|_{\ell_p} < \infty\},$$

where $m = (m(a_i, s_i, t_i))_{i\in\mathcal{I}}$. *Conversely, if* $(c_i(f))_{i\in\mathcal{I}} \in \ell_{p,m}$, *then* $f = \sum_{i\in\mathcal{I}} c_i \pi(a_i, s_i, t_i)\psi$ *is in* $\mathcal{SC}_{p,m}$ *and*

$$\|f\|_{\mathcal{SC}_{p,m}} \le C' \|(c_i(f))_{i\in\mathcal{I}}\|_{\ell_{p,m}}. \tag{16}$$

Given such an atomic decomposition, the problem arises under which conditions a function f is completely determined by its *moments* $\langle f, \pi(a_i, s_i, t_i)\psi\rangle$, $i \in \mathcal{I}$ and how f can be reconstructed from these moments. This is answered by the following theorem from [6] which establishes the existence of Banach frames.

Theorem 3.5. *Impose the same assumptions as in Theorem* 3.4. *Choose a neighborhood* U *of* e *such that*

$$\| \operatorname{osc}_{\psi,U} \|_{L_{1,w}(\mathbb{S})} < 1/\|\mathcal{SH}_\psi(\psi)\|_{L_{1,w}(\mathbb{S})}. \tag{17}$$

Then, for every U*-dense and relatively separated family* $X = \{(a_i, s_i, t_i) : i \in \mathcal{I}\}$ *in* $\mathbb{S}$ *the set* $\{\pi(a_i, s_i, t_i)\psi : i \in \mathcal{I}\}$ *is a* Banach frame *for* $\mathcal{SC}_{p,m}$. *This means that*

i) $f \in \mathcal{SC}_{p,m}$ *if and only if* $\left(\langle f, \pi(a_i, s_i, t_i)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}}\right)_{i\in\mathcal{I}} \in \ell_{p,m}$,

ii) *there exist two constants* $0 < D \le D' < \infty$ *independent on the choice of* p *such that*

$$D\,\|f\|_{\mathcal{SC}_{p,m}} \le \|(\langle f, \pi(a_i, s_i, t_i)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{i\in\mathcal{I}}\|_{\ell_{p,m}} \le D'\,\|f\|_{\mathcal{SC}_{p,m}}, \tag{18}$$

iii) *there exists a bounded, linear reconstruction operator* $\mathcal{R}$ *from* $\ell_{p,m}$ *to* $\mathcal{SC}_{p,m}$ *such that* $\mathcal{R}\left((\langle f, \psi(a_i, s_i, t_i)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{i\in\mathcal{I}}\right) = f$.

4. Structure of Shearlet Coorbit Spaces

4.1. **Atomic decomposition of Besov spaces.**

Let us recall the characterization of homogeneous Besov spaces $B^\sigma_{p,q}$ from [13], see also [16, 21]. For inhomogeneous Besov spaces we refer to [20]. For $\alpha > 1$, $D > 1$ and $K \in \mathbb{N}_0$, a K times differentiable function a on $\mathbb{R}^d$ is called a K*-atom* if the following two conditions are fulfilled:

A1) $\operatorname{supp} a \subset DQ_{j,m}(\mathbb{R}^d)$ for some $m \in \mathbb{R}^d$, where $Q_{j,m}(\mathbb{R}^d)$ denotes the cube in $\mathbb{R}^d$ centered at $\alpha^{-j}m$ with sides parallel to the coordinate axes and side length $2\alpha^{-j}$.

A2) $|D^\gamma a(x)| \le \alpha^{|\gamma| j}$ for $|\gamma| \le K$.

Now the homogeneous Besov spaces can be characterized as follows.

Theorem 4.1. *Let* $D > 1$ *and* $K \in \mathbb{N}_0$ *with* $K \ge 1 + \lfloor\sigma\rfloor$ *be fixed. Let* $1 \le p \le \infty$. *Then* $f \in B^\sigma_{p,q}$ *if and only if it can be represented* [1] *as*

$$f(x) = \sum_{j\in Z}\sum_{l\in\mathbb{Z}^d} \lambda(j,l) a_{j,l}(x), \tag{19}$$

where the $a_{j,l}$ *are* K*-atoms with* $\operatorname{supp} a_{j,l} \subset DQ_{j,l}(\mathbb{R}^d)$ *and*

$$\|f\|_{B^\sigma_{p,q}} \sim \inf\Big(\sum_{j\in\mathbb{Z}} \alpha^{j(\sigma-\frac{d}{p})q}\big(\sum_{l\in\mathbb{Z}^d} |\lambda(j,l)|^p\big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}$$

where the infimum is taken over all admissible representations (19).

[1] In the sense of distributions, a-posteriori implying norm convergence for $p < \infty$.

In this section, we are mainly interested in weights

$$m(a,s,t) = m(a) := |a|^{-r}, r \geq 0$$

and use the abbreviation

$$\mathcal{SC}_{p,r} := \mathcal{SC}_{p,m}.$$

For simplicity, we further assume in the following that we can use $\sigma = \tau = 1$ in the U-dense, relatively separated set (12) and restrict ourselves to the case $\epsilon = 1$. In other words, we assume that $f \in \mathcal{SC}_{p,r}$ can be written as

$$\begin{aligned} f(x) &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} c(j,k,l)\pi(\alpha^{-j}, \sigma\alpha^{-j/2}k, S_{\alpha^{-j/2}k}A_{\alpha^{-j}}l)\psi(x) \\ &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - \alpha^{j/2}kx_2 - l_1, \alpha^{j/2}x_2 - l_2). \end{aligned} \tag{20}$$

To derive reasonable trace and embedding theorems, it is necessary to introduce the following subspaces of $\mathcal{SC}_{p,r}$. For fixed $\psi \in B_w$ we denote by $\mathcal{SCC}_{p,r}$ the closed subspace of $\mathcal{SC}_{p,r}$ consisting of those functions which are representable as in (20) but with integers $|k| \leq \alpha^{j/2}$. As we shall see in the sequel for each of these ψ the resulting spaces $\mathcal{SCC}_{p,r}$ embed in the same scale of Besov spaces, and the same holds true for the trace theorems.

4.2. **A Density Result.** In most of the classical smoothness spaces like Sobolev and Besov spaces dense subsets of 'nice' functions can be identified. Typically, the set of Schwartz functions $\mathcal{S}$ serves as such a dense subset. We refer to [1] and any book of Hans Triebel for further information. By the following theorem the same is true for our shearlet coorbit spaces.

Theorem 4.2. *Let*

$$\mathcal{S}_0 := \left\{ f \in \mathcal{S} : |\hat{f}(\omega)| \leq \frac{\omega_1^{2\alpha}}{(1+\|\omega\|^2)^{2\alpha}} \; \forall \; \alpha > 0 \right\}$$

and $m(a,s,t) = m(a,s) := |a|^r(\frac{1}{|a|} + |a| + |s|)^n$ *for some* $r \in \mathbb{R}, n \geq 0$. *Then the set of Schwartz functions forms a dense subset of the shearlet coorbit space* $\mathcal{SC}_{p,m}$.

Proof. As in [6, Theorem 4.7] it can be shown that $\mathcal{S}_0$ is at least contained in $\mathcal{SC}_{p,m}$. (Note that in [6] the weight $(\frac{1}{|a|} + |a|)^r(\frac{1}{|a|} + |a| + |s|)^n$, $r, n > 0$ which is not smaller than 1 was considered.) It remains to show the density. To this end, we observe from Theorem 4.2 in [6] that certain band-limited Schwartz functions can be used as analyzing shearlets. Now let us recall that the atomic decomposition in (14) has to be understood as a limit of *finite* linear combinations with respect to the shearlet coorbit norm. However, every finite linear combination of Schwartz functions is again a Schwartz function, hence (14) implies that we have found for any $f \in \mathcal{SC}_{p,m}$ a sequence of Schwartz functions which converges to f. □

4.3. **Traces on the Real Axes.** In this subsection, we investigate the traces of functions lying in certain subspaces of $\mathcal{SC}_{p,r}$ with respect to the horizontal and vertical axes, respectively. With larger technical effort it is also possible to prove trace theorems with respect to more general lines. By $f \lesssim g$, the relation $f \leq C\, g$ with some constant $C \geq 0$ is meant.

Theorem 4.3. *Let* $Tr_h f$ *denote the restriction of* f *to the (horizontal)* x_1*-axis, i.e.,* $(Tr_h f)(x_1) := f(x_1, 0)$. *Then* $Tr_h(\mathcal{SCC}_{p,r}) \subset B^{\sigma_1}_{p,p}(\mathbb{R}) + B^{\sigma_2}_{p,p}(\mathbb{R})$, *where*

$$B^{\sigma_1}_{p,p}(\mathbb{R}) + B^{\sigma_2}_{p,p}(\mathbb{R}) := \{h \mid h = h_1 + h_2, h_1 \in B^{\sigma_1}_{p,p}(\mathbb{R}), h_2 \in B^{\sigma_2}_{p,p}(\mathbb{R})\}$$

and the parameters σ_1 and σ_2 satisfy the conditions

$$\sigma_1 = r - \frac{5}{4} + \frac{3}{2p}, \quad \sigma_2 = r - \frac{3}{4} + \frac{1}{p}.$$

Note that $\sigma_1 \leq \sigma_2$ for $p \geq 2$.

Proof. Using (20) we split f into $f = f_1 + f_2$, where

$$f_1(x_1,x_2) := \sum_{j\geq 0} \sum_{|k|\leq \alpha^{j/2}} \sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - \alpha^{j/2}kx_2 - l_1, \alpha^{j/2}x_2 - l_2), \tag{21}$$

$$f_2(x_1,x_2) := \sum_{j<0} \sum_{l\in\mathbb{Z}^2} c(j,0,l)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - l_1, \alpha^{j/2}x_2 - l_2). \tag{22}$$

By Theorem 3.3 we can choose ψ compactly supported in $[-D,D]\times[-D,D]$ for some $D > 1$. Moreover, we can assume that $|D_1^\gamma \psi| \leq 1$ for $0 \leq \gamma \leq K := \max\{K_1,K_2\}$, where $K_1 := 1 + \lfloor \sigma_1 \rfloor$, $K_2 := 1 + \lfloor \sigma_2 \rfloor$ and where $D_1\psi$ denotes the derivative with respect to the first component of ψ. Now $Tr_h f$ can be written as

$$\begin{aligned} Tr_h f(x_1) = f(x_1,0) &= \sum_{j\in\mathbb{Z}} \sum_{|k|\leq\alpha^{j/2}} \sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - l_1, -l_2) \\ &= \sum_{j\in\mathbb{Z}} \sum_{l_1\in\mathbb{Z}} \sum_{|k|\leq\alpha^{j/2}} \sum_{|l_2|\leq D} c(j,k,l_1,l_2)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - l_1, -l_2) \\ &= \sum_{j\geq 0} \sum_{l_1\in\mathbb{Z}} \lambda(j,l_1)a_{j,l_1}(x_1) + \sum_{j<0} \sum_{l_1\in\mathbb{Z}} \lambda(j,l_1)a_{j,l_1}(x_1) \\ &= Tr_h f_1(x_1) + Tr_h f_2(x_1), \end{aligned}$$

where for $j \geq 0$,

$$\begin{aligned} a_{j,l_1}(x_1) &:= \begin{cases} \lambda(j,l_1)^{-1}\alpha^{\frac{3}{4}j} \sum\limits_{|k|\leq\alpha^{j/2}} \sum\limits_{|l_2|\leq D} c(j,k,l_1,l_2)\psi(\alpha^j x_1 - l_1, -l_2) & \text{if } \lambda(j,l_1) \neq 0, \\ 0 & \text{otherwise}, \end{cases} \\ \lambda(j,l_1) &:= \alpha^{\frac{3}{4}j} \sum_{|k|\leq\alpha^{j/2}} \sum_{|l_2|\leq D} |c(j,k,l_1,l_2)|, \end{aligned}$$

and for $j < 0$

$$\begin{aligned} a_{j,l_1}(x_1) &:= \begin{cases} \lambda(j,l_1)^{-1}\alpha^{\frac{3}{4}j} \sum\limits_{|l_2|\leq D} c(j,0,l_1,l_2)\psi(\alpha^j x_1 - l_1, -l_2) & \text{if } \lambda(j,l_1) \neq 0, \\ 0 & \text{otherwise}, \end{cases} \\ \lambda(j,l_1) &:= \alpha^{\frac{3}{4}j} \sum_{|l_2|\leq D} |c(j,0,l_1,l_2)|. \end{aligned}$$

We have that $\operatorname{supp} \psi(\alpha^j x_1 - l_1, -l_2) \subset DQ_{j,l_1}(\mathbb{R})$ which is also true for all a_{j,l_1} and by construction we know further that $|D^\gamma a_{j,l_1}| \leq \alpha^{j\gamma}$, $0 \leq \gamma \leq K$. Thus, the a_{j,l_1} are K_1-atoms on $\mathbb{R}$. Next, we consider

$$\|Tr_h f_1\|_{B^{\sigma_1}_{p,p}} \lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{j(\sigma_1-\frac{1}{p})p} \sum_{l_1\in\mathbb{Z}} |\lambda(j,l_1)|^p\Big)^{\frac{1}{p}}$$
$$\lesssim \Big(\sum_{j\geq 0} \alpha^{jp(\sigma_1+\frac{3}{4}-\frac{1}{p})} \sum_{l_1\in\mathbb{Z}} \big(\sum_{|k|\leq \alpha^{j/2}} \sum_{|l_2|\leq D} |c(j,k,l_1,l_2)|\big)^p\Big)^{\frac{1}{p}}.$$

Since $\big(\sum_{i=1}^N |z_i|\big)^p \leq N^{p-1}\sum_{i=1}^N |z_i|^p$ and the set $\{k\in\mathbb{Z} : |k|\leq \alpha^{j/2}\}$ contains $C\alpha^{j/2}$ elements we can estimate

$$\|Tr_h f_1\|_{B^{\sigma_1}_{p,p}} \lesssim \Big(\sum_{j\geq 0} \alpha^{jp(\sigma_1+\frac{5}{4}-\frac{3}{2p})} \sum_{|k|\leq \alpha^{j/2}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p\Big)^{\frac{1}{p}}$$
$$\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p\Big)^{\frac{1}{p}} \lesssim \|f\|_{\mathcal{SC}_{p,r}}$$

with $r = \sigma_1 + \frac{5}{4} - \frac{3}{2p}$. In the same way we obtain that

$$\|Tr_h f_2\|_{B^{\sigma_2}_{p,p}} \lesssim \Big(\sum_{j<0} \alpha^{jp(\sigma_2+\frac{3}{4}-\frac{1}{p})} \sum_{l\in\mathbb{R}^2} |c(j,0,l)|^p\Big)^{\frac{1}{p}}$$
$$\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p\Big)^{\frac{1}{p}} \lesssim \|f\|_{\mathcal{SC}_{p,r}}$$

with $r = \sigma_2 + \frac{3}{4} - \frac{1}{p}$. This completes the proof. □

By the following corollary the restriction to $\mathcal{SCC}_{p,r}$ is not necessary for $p = 1$.

Corollary 4.4. *For $p = 1$, the embedding $Tr_h(\mathcal{SC}_{1,r}) \subset B^{\sigma}_{1,1}(\mathbb{R})$ with $\sigma = r - \frac{3}{4} + \frac{1}{p}$ holds true.*

Proof. Following the lines of the previous proof, where the summation with respect to k is over $\mathbb{Z}$, we obtain

$$\|Tr_h f\|_{B^{\sigma}_{1,1}} \lesssim \sum_{j\in\mathbb{Z}} \alpha^{j((\sigma+\frac{3}{4})p-1)} \sum_{l_1\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{|l_2|\leq D} |c(j,k,l_1,l_2)| \leq C\|f\|_{\mathcal{SC}_{1,r}}$$

with $r = \sigma + \frac{3}{4} - \frac{1}{p}$ and we are done. □

Let us turn to traces on the vertical axis.

Theorem 4.5. *Let $Tr_v f$ denote the restriction of f to the (vertical) x_2-axis, i.e., $(Tr_v f)(x_2) := f(0,x_2)$. Then the embedding $Tr_v(\mathcal{SCC}_{p,r}) \subset B^{\sigma_1}_{p,p}(\mathbb{R}) + B^{\sigma_2}_{p,p}(\mathbb{R})$, holds true, where σ_1 is the largest number such that*

$$\sigma_1 + \lfloor\sigma_1\rfloor \leq 2r - \frac{9}{2} + \frac{3}{p}, \quad \text{and} \quad \sigma_2 = 2r - \frac{3}{2} + \frac{1}{p}.$$

Proof. As in (21) and(21) we split f into $f = f_1 + f_2$, where we can choose ψ compactly supported in $[-D,D]\times[-D,D]$ for some $D > 1$ and normalized such that the derivatives of order $0 \leq \gamma \leq K$ with $K := \max\{K_1, K_2\}$, where $K_1 := 1 + \lfloor\sigma_1\rfloor$, $K_2 := 1 + \lfloor\sigma_2\rfloor$ are not larger than 1. By the support assumption on ψ we have that

$$\alpha^{-j/2}(l_2 - D) \;\leq\; x_2 \;\leq\; \alpha^{-j/2}(l_2 + D),$$
$$-kl_2 - D(1+|k|) \;\leq\; l_1 \;\leq\; -kl_2 + D(1+|k|).$$

Let $I_{k,l_2} := \{r \in \mathbb{Z} : |r + kl_2| \le D(1+|k|)\}$. Now we obtain that

$$Tr_v f(x_2) = f(0,x_2) = \sum_{j\in\mathbb{Z}} \sum_{|k|\le\alpha^{j/2}} \sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j}\psi(-\alpha^{j/2}kx_2 - l_1, \alpha^{j/2}x_2 - l_2).$$

This can be rewritten as

$$\begin{aligned} f(0,x_2) &= \sum_{j\ge 0}\sum_{l_2\in\mathbb{Z}} \lambda(j,l_2)a_{j,l_2}(x_2) + \sum_{j<0}\sum_{l_2\in\mathbb{Z}} \lambda(j,l_2)a_{j,l_2}(x_2) \\ &= Tr_v f_1(x_2) + Tr_v f_2(x_2), \end{aligned}$$

where for $j \ge 0$,

$$a_{j,l_2}(x_2) := \lambda(j,l_2)^{-1}\alpha^{\frac{3+2K_1}{4}j} \sum_{|k|\le\alpha^{j/2}} \sum_{l_1\in I_{k,l_2}} c(j,k,l_1,l_2)\alpha^{-K_1 j/2}\psi(-\alpha^{j/2}kx_2 - l_1, \alpha^{j/2}x_2 - l_2)$$

if $\lambda(j,l_2) \neq 0$ and $a_{j,l_2}(x_2) = 0$ otherwise and

$$\lambda(j,l_2) := \alpha^{\frac{3+2K_1}{4}j} \sum_{|k|\le\alpha^{j/2}} \sum_{l_1\in I_{k,l_2}} |c(j,k,l_1,l_2)|$$

and for $j < 0$,

$$a_{j,l_2}(x_2) := \lambda(j,l_2)^{-1}\alpha^{\frac{3}{4}j} \sum_{|l_1|\le D} c(j,0,l_1,l_2)\psi(-l_1, \alpha^{j/2}x_2 - l_2)$$

if $\lambda(j,l_2) \neq 0$ and $a_{j,l_2}(x_2) = 0$ otherwise and

$$\lambda(j,l_2) := \alpha^{\frac{3}{4}j} \sum_{|l_1|\le D} |c(j,0,l_1,l_2)|.$$

We have that $\operatorname{supp}\psi(-\alpha^{j/2}kx_2 - l_1, \alpha^{j/2}x_2 - l_2) \subset DQ_{j,l_2}(\mathbb{R})$, where the cube is considered with respect to $\sqrt{\alpha}$ now. This is also true for a_{j,l_2}. For $j \ge 0$ we conclude by $|k| \le \alpha^{j/2}$ that $\alpha^{-K_1 j/2}|D^\gamma\psi(-\alpha^{j/2}kx_2 - l_1, \alpha^{j/2}x_2 - l_2)| \le \alpha^{\frac{j}{2}\gamma}$ and consequently $|D^\gamma a_{j,l_2}| \le \alpha^{\frac{j}{2}\gamma}$, $\gamma \le K_1$. For $j < 0$ we also have that $|D^\gamma a_{j,l_2}| \le \alpha^{\frac{j}{2}\gamma}$. Thus a_{j,l_2} are K_1-atoms. We get

$$\begin{aligned} \|Tr_v f_1\|_{B^{\sigma_1}_{p,p}} &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_1 - \frac{1}{p})p} \sum_{l_2\in\mathbb{Z}} |\lambda(j,l_2)|^p\Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{j\ge 0} \alpha^{\frac{j}{2}(\sigma_1 - \frac{1}{p})p}\alpha^{\frac{j}{2}(\frac{3+2K_1}{2})p}\alpha^{\frac{j}{2}(2-\frac{2}{p})p} \sum_{|k|\le\alpha^{j/2}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p\Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_1 + \frac{7}{2} + K_1 - \frac{3}{p})p} \sum_{|k|\le\alpha^{j/2}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p\Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_1 + \frac{7}{2} + 1 + \lfloor\sigma_1\rfloor - \frac{3}{p})p} \sum_{|k|\le\alpha^{j/2}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p\Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p\Big)^{\frac{1}{p}} \\ &\lesssim \|f\|_{\mathcal{SC}_{p,r}} \end{aligned}$$

with $r \geq \frac{1}{2}(\sigma_1 + \lfloor \sigma_1 \rfloor + \frac{9}{2} - \frac{3}{p})$. Analogously we can compute

$$\begin{aligned} \|Tr_v f_2\|_{B^{\sigma_2}_{p,p}} &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_2 - \frac{1}{p})p} \sum_{l_2 \in \mathbb{Z}} |\lambda(j,l_2)|^p \Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{j<0} \alpha^{\frac{j}{2}(\sigma_2 - \frac{1}{p} + \frac{3}{2})p} \sum_{l \in \mathbb{R}^2} |c(j,0,l)|^p \Big)^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}} \sum_{l \in \mathbb{R}^2} |c(j,k,l)|^p \Big)^{\frac{1}{p}} \\ &\lesssim \|f\|_{\mathcal{SC}_{p,r}} \end{aligned}$$

with $r = \frac{1}{2}(\sigma_2 + \frac{3}{2} - \frac{1}{p})$ and we are done. □

4.4. **Embedding Results.** In this subsection, we prove embedding results of certain subspaces of shearlet coorbit spaces into (sums of) homogeneous Besov spaces. But first we provide a result concerning the embedding within shearlet coorbit spaces. In [9, Section 5.7] some embedding theorems for general $L_{p,m}$ coorbit spaces were given. In particular, the authors mentioned that for a fixed weight m, these spaces are monotonically increasing with p. The following corollary is a special results in this direction.

Corollary 4.6. *For $1 \leq p_1 \leq p_2 \leq \infty$ the embedding $\mathcal{SC}_{p_1,r} \subset \mathcal{SC}_{p_2,r}$ holds true. Introducing the 'smoothness spaces' $\mathcal{G}^r_p := \mathcal{SC}_{p,r+d(\frac{1}{2}-\frac{1}{p})}$. this implies the continuous embedding*

$$\mathcal{G}^{r_1}_{p_1} \subset \mathcal{G}^{r_2}_{p_2}, \quad \text{if} \quad r_1 - \frac{d}{p_1} = r_2 - \frac{d}{p_2}.$$

For convenience we add the simple proof.

Proof. By Theorem 3.4 we obtain that

$$\|f\|_{\mathcal{SC}_{p_2,r}} \lesssim \|(c_\epsilon(j,k,l)\|_{\ell_{p_2,r}} \lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jrp_2} \sum_{\substack{k,l \\ \epsilon\in\{-1,1\}}} |c_\epsilon(j,k,l)|^{p_2} \Big)^{\frac{1}{p_2}},$$

where $c_\epsilon(j,k,l)$ is the coefficient belonging in the representation (14) with respect to (12) to the function $\pi(\epsilon\alpha^{-j}, \sigma\alpha^{-j/2}k, S_{\sigma\alpha^{-j/2}k} A_{\alpha^{-j}} \tau l)\psi$. Since $\ell_{p_1} \subset \ell_{p_2}$ for $p_1 \leq p_2$ we get finally that

$$\begin{aligned} \|f\|_{\mathcal{SC}_{p_2,r}} &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jrp_2} \big(\sum_{\substack{k,l \\ \epsilon\in\{-1,1\}}} |c_\epsilon(j,k,l)|^{p_1} \big)^{\frac{p_2}{p_1}} \Big)^{\frac{1}{p_2}} \\ &\lesssim \Big(\sum_{j\in\mathbb{Z}} \alpha^{jrp_1} \sum_{\substack{k,l \\ \epsilon\in\{-1,1\}}} |c_\epsilon(j,k,l)|^{p_1} \Big)^{\frac{1}{p_1}} \lesssim \|f\|_{\mathcal{SC}_{p_1,r}}. \end{aligned}$$

□

Next we state our final result.

Theorem 4.7. *The embedding $\mathcal{SCC}_{p,r} \subset B^{\sigma_1}_{p,p}(\mathbb{R}^2) + B^{\sigma_2}_{p,p}(\mathbb{R}^2)$, holds true, where σ_1 is the largest number such that*

$$\sigma_1 + \lfloor \sigma_1 \rfloor \leq 2r - \frac{9}{2} + \frac{4}{p}, \quad \text{and} \quad \sigma_2 - \frac{\lfloor \sigma_2 \rfloor}{2} = r + \frac{3}{2p} + \frac{1}{4}.$$

Proof. By (20) we know that $f \in \mathcal{SCC}_{p,r}$ can be written as

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{|k|\le\alpha^{j/2}} \sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - \alpha^{j/2}kx_2 - l_1, \alpha^{j/2}x_2 - l_2),$$

where we can choose ψ compactly supported in $[-D,D]\times[-D,D]$ for some $D>1$ and normalized such that the derivatives of order $0\le|\gamma|\le K := \max\{K_1,K_2\},\ K_1 := 1+\lfloor\sigma_1\rfloor,\ K_2 := 1+\lfloor\sigma_2\rfloor$ are not larger than 1.

We split $f\in\mathcal{SCC}_{p,r}$ as in (21) and (22) into f_1 and f_2. Then we obtain with the index transform $l_1 = r_1 - kl_2$ that

$$\begin{aligned} f_1(x) = &\sum_{j\ge0} \sum_{|k|\le\alpha^{j/2}} \sum_{l_2\in\mathbb{Z}} \sum_{n_1\in\mathbb{Z}} \sum_{r_1\in I(j,n_1)} c(j,k,r_1-kl_2,l_2)\alpha^{\frac{3}{4}j} \\ &\times\psi(\alpha^j x_1 - \alpha^{j/2}kx_2 - r_1 + kl_2, \alpha^{j/2}x_2 - l_2) \end{aligned}$$

where $I(j,n_1) := \{r\in\mathbb{Z} : \alpha^{j/2}(n_1-1) < r \le \alpha^{j/2}n_1\}$.

For $j\ge0$ we set

$$\begin{aligned} a_{j,n_1,l_2}(x) := &\lambda(j,n_1,l_2)^{-1}\alpha^{\frac{3+2K_1}{4}j} \sum_{|k|\le\alpha^{j/2}} \sum_{r_1\in I(j,n_1)} c(j,k,r_1-kl_2,l_2) \\ &\times\alpha^{-K_1 j/2}\,\psi(\alpha^j x_1 - \alpha^{j/2}kx_2 - r_1 + kl_2, \alpha^{j/2}x_2 - l_2), \end{aligned}$$

if $\lambda(j,n_1,l_2)\ne0$ and $a_{j,n_1,l_2}(x)=0$ otherwise, where

$$\lambda(j,n_1,l_2) := \alpha^{\frac{3+2K_1}{4}j} \sum_{|k|\le\alpha^{j/2}} \sum_{r_1\in I(j,n_1)} |c(j,k,r_1-kl_2,l_2)|.$$

By the support assumption on ψ, the functions appearing in the definition of a_{j,n_1,m_2} are only non-zero if the following conditions are satisfied:

$$-D \le \alpha^{j/2}x_2 - l_2 \le D, \qquad \alpha^{-j/2}(l_2 - D) \le x_2 \le \alpha^{-j/2}(l_2+D)$$

and

$$-D \le \alpha^j x_1 - \alpha^{j/2}kx_2 - r_1 + kl_2 \le D,$$

$$\begin{aligned} \alpha^{-j}r_1 + \alpha^{-j}k(\alpha^{j/2}x_2 - l_2) - \alpha^{-j}D &\le x_1 \le \alpha^{-j}r_1 + \alpha^{-j}k(\alpha^{j/2}x_2 - l_2) + \alpha^{-j}D, \\ \alpha^{-j}r_1 - \alpha^{-j/2}(2D) &\le x_1 \le \alpha^{-j}r_1 + \alpha^{-j/2}(2D), \\ \alpha^{-j/2}n_1 - \alpha^{-j/2}(3D) &\le x_1 \le \alpha^{-j/2}n_1 + \alpha^{-j/2}(2D). \end{aligned}$$

Thus, a_{j,n_1,l_2} is supported in $3DQ_{j,n_1,l_2}$, where the cube is considered with respect to $\sqrt{\alpha}$. The appropriate bounds $|D^\gamma a_{j,n_1,l_2}| \le \alpha^{\frac{j}{2}|\gamma|}$, $|\gamma|\le K_1$ can be derived as in the previous proof. Hence the functions a_{j,n_1,l_2} are K_1-atoms.

Now we obtain for

$$f_1(x) = \sum_{j\ge0}\sum_{l_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}} \lambda(j,n_1,l_2)a_{j,n_1,l_2}(x)$$

that

$$
\begin{aligned}
\|f_1\|^p_{B^{\sigma_1}_{p,p}} &\lesssim \sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_1-\frac{2}{p})p} \sum_{l_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}} |\lambda(j,n_1,l_2)|^p \\
&\lesssim \sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_1-\frac{2}{p})p} \alpha^{\frac{j}{2}(\frac{3+2K_1}{2})p} \sum_{l_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}} \left| \sum_{|k|\le\alpha^{j/2}} \sum_{r_1\in I(j,n_1)} |c(j,k,r_1-kl_2,l_2)| \right|^p \\
&\lesssim \sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}p(\sigma_1+\frac{7}{2}+K_1-\frac{4}{p})} \sum_{l_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}} \sum_{|k|\le\alpha^{j/2}} \sum_{r_1\in I(j,n_1)} |c(j,k,r_1-kl_2,l_2)|^p \\
&\lesssim \sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}p(\sigma_1+\frac{9}{2}+\lfloor\sigma_1\rfloor-\frac{4}{p})} \sum_{|k|\le\alpha^{j/2}} \sum_{l_1\in\mathbb{Z}}\sum_{l_2\in\mathbb{Z}} |c(j,k,l_1,l_2)|^p \\
&\lesssim \|f\|^p_{\mathcal{SC}_{p,r}}.
\end{aligned}
$$

In the case $j<0$ we obtain with $J(j,n_2) := \{r : \alpha^{-j/2}(n_2-1) < r \le \alpha^{-j/2}n_2\}$ that

$$
\begin{aligned}
f_2(x) &= \sum_{j<0}\sum_{l_1\in\mathbb{Z}}\sum_{l_2\in\mathbb{Z}} c(j,0,l_1,l_2)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - l_1, \alpha^{j/2}x_2 - l_2) \\
&= \sum_{j<0}\sum_{l_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{r_2\in J(j,n_2)} c(j,0,l_1,r_2)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - l_1, \alpha^{j/2}x_2 - r_2) \\
&= \sum_{j<0}\sum_{l_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} \lambda(j,l_1,n_2) a_{j,l_1,n_2}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{j,l_1,n_2}(x) &:= \lambda(j,l_1,n_2)^{-1}\alpha^{\frac{3-2K_2}{4}j} \sum_{r_2\in J(j,n_2)} c(j,0,l_1,r_2)\alpha^{\frac{jK_2}{2}}\psi(\alpha^j x_1 - l_1, \alpha^{j/2}x_2 - r_2), \\
\lambda(j,l_1,n_2) &:= \alpha^{\frac{3-2K_2}{4}j} \sum_{r_2\in J(j,n_2)} |c(j,0,l_1,r_2)|
\end{aligned}
$$

and $a_{j,l_1,n_2}(x) := 0$ if $\lambda_{j,l_1,n_2} = 0$. By the support assumption on ψ we get

$$
\begin{aligned}
\alpha^{-j}(l_1-D) &\le x_1 \le \alpha^{-j}(l_1+D), \\
\alpha^{-j/2}(r_2-D) &\le x_2 \le \alpha^{-j/2}(r_2+D) \quad \Rightarrow \quad \alpha^{-j}(n_2-2D) \le x_2 \le \alpha^{-j}(n_2+D).
\end{aligned}
$$

Consequently, a_{j,l_1,n_2} is supported in $2DQ_{j,l_1,n_2}$. Since $1 \ge \alpha^{j|\gamma|/2} \ge \alpha^{j|\gamma|} \ge \alpha^{jK_2}$ for $0 \le |\gamma| \le K_2$ and $j<0$ we obtain further that $|D^\gamma a_{j,n_1,l_2}| \le \alpha^{jK_2/2}\alpha^{j|\gamma|/2} \le \alpha^{j|\gamma|}$ so that a_{j,l_1,n_2} are K_2-atoms..

Thus,

$$\begin{aligned}
\|f_2\|^p_{B^{\sigma_2}_{p,p}} &\lesssim \sum_{j\in\mathbb{Z}} \alpha^{j(\sigma_2-\frac{2}{p})p} \sum_{l_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} |\lambda(j,l_1,n_2)|^p \\
&\lesssim \sum_{j<0} \alpha^{j(\sigma_2-\frac{2}{p}+\frac{3-2K_2}{4})p} \sum_{l_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} \left| \sum_{r_2\in J(j,n_2)} c(j,0,l_1,r_2) \right|^p \\
&\lesssim \sum_{j<0} \alpha^{j(\sigma_2-\frac{3}{2p}+\frac{1}{4}-\frac{K_2}{2})p} \sum_{l\in\mathbb{R}^2} |c(j,0,l)|^p \\
&\lesssim \sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{R}^2} |c(j,k,l)|^p \\
&\lesssim \|f\|^p_{\mathcal{SC}_{p,r}},
\end{aligned}$$

where $r = \sigma_2 - \frac{3}{2p} - \frac{1}{4} - \frac{\lfloor\sigma_2\rfloor}{2}$. □

Remark 4.8. An alternative way to obtain trace results would be first to apply the Besov embedding and afterwards the classical trace theorem for homogeneous Besov spaces. Let us briefly discuss the relation between these different approaches. For simplicity we restrict ourselves to the positives scales and traces to the x_2-axis. Usually an application of trace theorems in Besov spaces leads to a loss of smoothness of order $1/p$, that is $\mathrm{Tr}(B^s_{pp}(\mathbb{R}^d)) = B^{s-1/p}_{pp}(\mathbb{R}^{d-1})$, see [13]. Let the coorbit space smoothness index r be fixed. Depending on the concrete values of r and p, the direct and the indirect approach can yield the same result. However, in specific cases it turns out that the direct approach is superior as we gain some smoothness: Let $2r - \frac{9}{2} + \frac{3}{p} = 2a + \alpha$ with $a \in \mathbb{Z}$ and $\alpha \in [0,2)$. Then we have for $\alpha \in [0,1)$ by Theorem 4.5 that $\sigma_1 = a + \alpha$. On the other hand, in case $\alpha + \frac{1}{p} \in [1,2)$ an application of Theorem 4.7 yields $\mathcal{S}CC_{p,r} \subset B^{\tilde\sigma_1}_{pp}$, where $\tilde\sigma_1 = a + 1 - \varepsilon$ for arbitrary small $\varepsilon > 0$. Consequently, applying the trace theorem for Besov spaces yields smoothness $\tilde\sigma_1 - 1/p = a + 1 - \varepsilon - 1/p < a + \alpha = \sigma_1$.

Remark 4.9. Embedding results in Besov spaces have also been shown for the curvelet setting by Borup and Nielsen [4]. However, the technique used by these authors is completely different. In contrast to our approach they work in the frequency domain. We prefer to consider the time domain with flexible atomic decompositions for the following reasons. As already outlined above time domain techniques provide a very natural way to derive trace theorems which might be very difficult or even impossible in the Fourier domain. Moreover, since we are working with compactly supported atoms the treatment of shearlet coorbit spaces on bounded domains, including again embedding and trace theorems, seems to be manageable.

We also think that our approach provides some advantages for higher dimensions. Trace theorems for shearlet coorbit spaces on $\mathbb{R}^n$, $n \geq 3$ to higher dimensional hyperplanes are not straightforward since it is not clear that these traces will also be contained in Besov spaces. One natural conjecture would be that the traces of shearlet coorbit spaces on $\mathbb{R}^3$ with respect to two-dimensional hyperplanes are again shearlet coorbit spaces. To prove this conjecture, again some kind of flexible atomic decomposition techniques for shearlet coorbit spaces would be needed which is not available up to now. However, by following the lines corresponding to the Besov space setting there is some hope that such flexible decompositions can be derived. These questions will be discussed in forthcoming papers.

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