Weighted Coorbit Spaces and Banach Frames on Homogeneous Spaces

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Abstract

This paper is concerned with frame constructions on domains and manifolds. The starting point is a unitary group representation which is square integrable modulo a suitable subgroup and therefore gives rise to a generalized continuous wavelet transform. Then generalized coorbit spaces can be defined by collecting all functions for which this wavelet transform is contained in a weighted L_p–space. Moreover, we show that a judicious discretization of the representation leads to an atomic decomposition and to Banach frames for these coorbit spaces.

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1 Introduction

One of the classical tasks in applied analysis is the efficient representation/analysis of a given signal. Usually, the first step is the decomposition of the signal into suitable building blocks. Starting with Fourier analysis around 1820, many more or less successful approaches have been suggested. Current interest especially centers around multiscale representations of wavelet type. Wavelet bases have several remarkable advantages. Among others, they give rise to characterizations of function spaces such as Besov spaces and provide powerful approximation schemes, see, e.g., [5, 6]. However, in recent studies, it has turned out that the use of Riesz bases may have some serious drawbacks. One important problem is the lack of flexibility which is in some sense a consequence of the uniqueness of the representation. Therefore, one natural way out suggests itself: why not using a slightly weaker concept and allowing some redundancies, i.e., why not working with frames? In general, given a Hilbert space $\mathcal{H}$, a collection of elements $\{e_i\}_{i\in\mathbb{Z}}$ is called a *frame* if there exist constants $0 < A_1 \leq A_2 < \infty$ such that

$$A_1\|f\|_{\mathcal{H}}^2 \leq \sum_{i\in\mathbb{Z}} |\langle f, e_i\rangle_{\mathcal{H}}|^2 \leq A_2\|f\|_{\mathcal{H}}^2. \tag{1.1}$$

The frame concept has been introduced by Duffin and Schäfer [7] in 1952. However, the starting point of the modern frame theory was the fundamental Feichtinger/Gröchenig theory which has been developed since 1986 in a series of papers [9, 10, 11, 12, 13]. This very aestetic and subtle theory is essentially based on group theory. Given a Hilbert space $\mathcal{H}$, the first step is to find a suitable group $\mathcal{G}$ that admits a (square) integrable representation in $\mathcal{H}$ and therefore gives rise to a generalized (continuous) wavelet transform. Then, so–called *coorbit spaces* can be defined by collecting all functions for which this wavelet transform is contained in some weighted L_p–space. Finally, a judicious discretization of the representation produces the desired frames for the coorbit spaces. This approach works fine for the whole Euclidean plane and produces a general framework that covers, e.g., the classical wavelet and Weyl–Heisenberg frames. However, when it comes to practical applications, also the case of bounded domains and manifolds is important. Then, very often the problem arises that the group acting on the manifold is too 'large', i.e., its representation is not square integrable. One natural remedy as suggested, e.g., by Ali et al. [1] and Torresani [18], is the concept of square–integrability modulo quotients. In this case, one has to find a certain subgroup $\mathcal{P}$ such that, after restricting the representation to the induced quotient space $\mathcal{G}/\mathcal{P}$, one is again in a square integrable setting. However, by this passage to quotients the very convenient group structure gets lost, so that many of the building blocks used in the Feichtinger/Gröchenig theory such as convolutions are no longer available. Nevertheless, in the previous paper [3], we have shown that a quite natural generalization of the Feichtinger/Gröchenig theory to quotient spaces is indeed possible. The major tool was a generalized reproducing kernel. The application of the corresponding integral operator in some sense replaces the usual convolution. Then, under certain integrability conditions on this kernel it has turned out that all the basic steps of the Feichtinger/Gröchenig approach can still be performed. By employing the concept of square–integrability modulo quotients, generalized coorbit spaces may be defined. Moreover, one can define an approximation operator which produces atomic decompositions for these coorbit spaces. Furthermore, a reconstruction operator can be introduced in a similar fashion and the Banach frame property can be established.

To keep the technical difficulties at a reasonable level, in [3] only the 'simplest' class of coorbit spaces was considered. However, the coorbit approach allows the definition of whole scales of smoothness spaces by collecting all functions for which the generalized wavelet transform has certain decay properties, i.e., by considering *weighted* spaces. To fill this gap is the major aim of the present work.

This paper is organized as follows. In Section 2, we collect all the facts on group theory that are needed for our purposes. Then, in Section 3, we introduce and analyze our generalized weighted coorbit spaces. Section 4 contains the main results of this paper. In Subsection 4.1 we explain the setting and state all the conditions that are needed to establish atomic decompositions and Banach frames for the generalized weighted coorbit spaces. Subsection 4.2 is devoted to the definition and the analysis of the underlying approximation operators. Then, in Subsection 4.3 we establish the frame bounds. This part of our analysis is essentially based on a version of the Riesz–Thorin interpolation theorem. Since this specific version was not found in the literature, we have included a proof based on complex interpolation in the appendix. It turns out that the analysis can also be carried over to the dual spaces of the weighted coorbit spaces. The arguments are sketched in Subsection 4.4. Finally, in Section 5, we exlain how the whole machinery can be used to analyze functions on the spheres. In particular, we show that the analysis presented in this paper enables us to define generalized modulation spaces on the spheres and to construct atomic decompositions and Banach frames for them.

2 Group Theoretical Background

Let $\mathcal{G}$ be a separable, locally compact, topological Hausdorff group with right Haar measure ν. A *unitary representation* of $\mathcal{G}$ in a Hilbert space $\mathcal{H}$ is defined as a mapping U of $\mathcal{G}$ into the space of unitary operators on $\mathcal{H}$ such that $U(g \circ g') = U(g)U(g')$ for all $g, g' \in \mathcal{G}$ and $U(e) = Id$. A unitary representation U is called

- *irreducible*, if the only closed subspaces of $\mathcal{H}$ which are invariant under all operators $U(g)$ $(g \in \mathcal{G})$ are $\{0\}$ and $\mathcal{H}$,
- *strongly continuous*, if the mapping $g \mapsto U(g)\varphi$ is continuous from $\mathcal{G}$ to $\mathcal{H}$ for all $\varphi \in \mathcal{H}$.

The representation U is said to be *square integrable*, if there exists a nonzero vector $\psi \in \mathcal{H}$ which fulfills the admissibility condition

$$\int_{\mathcal{G}} |\langle \psi, U(g)\psi \rangle_{\mathcal{H}}|^2 \, d\nu(g) < \infty \ .$$

Strongly continuous, irreducible, unitary representations which are square integrable form the background of the short–time Fourier transform and the continuous wavelet transform, where the relevant groups are the reduced Weyl–Heisenberg group and the affine group, respectively. Unfortunately, there are many cases of practical interest, where the group is too large such that no square integrable representation exists. Typical examples are the Schrödinger representations of the Weyl-Heisenberg group on $L_2(\mathbb{R}^n)$. Very often, these cases can be handled by restricting U to a homogeneous space $X = \mathcal{G}/\mathcal{P}$, where $\mathcal{P}$ is a closed subgroup of $\mathcal{G}$. Unless otherwise stated, we shall always consider right coset spaces $\mathcal{P}g$ ($g \in \mathcal{G}$). Because U is not directly defined on X, it is necessary to embed X in $\mathcal{G}$. This can be realized by using the canonical fiber bundle structure of $\mathcal{G}$ with projection $\Pi : \mathcal{G} \to X$. In the following, let $\sigma : X \to \mathcal{G}$ be a Borel section of this fiber bundle, i.e., $\Pi \circ \sigma(h) = h$ for all $h \in X$. In this paper, we always assume that X carries a $\mathcal{G}$–invariant measure μ, i.e., a measure invariant under the action $h \mapsto hg$ ($h \in X, g \in \mathcal{G}$). Of course such a measure does not exist for every $X = \mathcal{G}/\mathcal{P}$. However, X always carries a quasi–invariant measure and possible generalizations to this setting will be studied in a forthcoming paper [4]. Unless otherwise stated, in this paper $\langle \cdot, \cdot \rangle$ always denotes the L_2–inner product

$$\langle F, G \rangle = \int_X F(x)\overline{G(x)} \, d\mu(x)$$

whenever the integral is defined.

By [1], an irreducible, unitary representation U of $\mathcal{G}$ on $\mathcal{H}$ is called *square integrable* mod $(\mathcal{P}, \sigma)$, if there exists $\psi \in \mathcal{H}$ such that the integral

$$\int_X \langle f, U(\sigma(h)^{-1})\psi \rangle_{\mathcal{H}} \, U(\sigma(h)^{-1})\psi \, d\mu(h)$$

converges weakly to a positive, bounded operator A_σ (dependent on σ and ψ) which has a bounded inverse A_σ^{-1}, in the sense that

$$\langle A_\sigma f, g \rangle_{\mathcal{H}} = \int_X \langle f, U(\sigma(h)^{-1})\psi \rangle_{\mathcal{H}} \, \overline{\langle g, U(\sigma(h)^{-1})\psi \rangle_{\mathcal{H}}} \, d\mu(h). \tag{2.1}$$

If $A_\sigma = \lambda \, Id$ for some $\lambda > 0$, then we call U *strictly square integrable* mod $(\mathcal{P}, \sigma)$ and (ψ, σ) a *strictly admissible pair*. In this paper, we focus our attention to strictly square integrable representations, where we normalize ψ so that $\lambda = 1$. The general square integrable case for homogeneous spaces will be handled in the forthcoming paper [4].

If U is strictly square integrable mod $(\mathcal{P},\sigma)$ for $\psi \in \mathcal{H}$, then it is well–known, see, e.g., [1] that the set

$$\mathcal{O}_\sigma := \{U(\sigma(h)^{-1})\psi : h \in X\}$$

is total in $\mathcal{H}$, i.e., $(\mathcal{O}_\sigma)^\perp = \{0\}$ and that the map $V_\psi : \mathcal{H} \to L_2(X)$ given by

$$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi\rangle_{\mathcal{H}} \tag{2.2}$$

is an isometry from $\mathcal{H}$ onto the reproducing kernel Hilbert space

$$\mathcal{M}_2 := \{F \in L_2(X) : \ \langle F, R(h,\cdot)\rangle = F(h)\}$$

with Hermitian reproducing kernel

$$\begin{aligned} R(h,l) \ = \ R_{(\psi,\sigma)}(h,l) \ &:= \ \langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1})\psi\rangle_{\mathcal{H}} \\ &= \ \langle \psi, U(\sigma(h)\sigma(l)^{-1})\psi\rangle_{\mathcal{H}} \\ &= \ V_\psi(U(\sigma(h)^{-1})\psi)(l). \end{aligned} \tag{2.3}$$

Note that by the Schwarz inequality

$$\operatorname*{ess\ sup}_{h,l\in X} |R(h,l)| \le \|\psi\|^2_{\mathcal{H}}. \tag{2.4}$$

Thus, V_ψ can be inverted on its range $\mathcal{M}_2$ by its adjoint V_ψ^* given by

$$V_\psi^* F(s) := \int_X F(h) U(\sigma(h)^{-1})\psi(s)\, d\mu(h).$$

For $f \in \mathcal{H}$, this provides us with the reconstruction formula

$$f = V_\psi^* V_\psi f = \int_X \langle f, U(\sigma(h)^{-1})\psi\rangle_{\mathcal{H}} U(\sigma(h)^{-1})\psi\, d\mu(h). \tag{2.5}$$

3 Weighted Coorbit Spaces on Homogeneous Spaces

In this section, we extend our considerations of functions belonging to coorbit spaces on manifolds, cf. [3], to the concept of weighted coorbit spaces. By this extention we are able to characterize a wide range of function spaces on manifolds, e.g., in dependence on the underlying group we may obtain generalized modulation and Besov spaces, respectively, or some mixed function spaces. In order to keep comparisons as simple as possible, we adapt the notations given in [3, 9, 10, 11, 12, 13].

Let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{P},\sigma)$ with a strictly admissible function ψ. We introduce a positive, continuous weight function w on $\mathcal{G}$ which is in addition submultiplicative, i.e., $w(g \circ \tilde{g}) \le w(g)\, w(\tilde{g})$ for all $g, \tilde{g} \in \mathcal{G}$, and uniformly bounded from below, i.e., $w(g) \geq 1$ for all $g \in \mathcal{G}$. Associated with w we are concerned with the weighted L_p–spaces on $X = \mathcal{G}/\mathcal{P}$ defined for $1 \leq p < \infty$ by

$$L_{p,w}(X) := \{f \text{ measurable on } X : \|f\|_{L_{p,w}} := \left(\int_X |f(h)|^p w(\sigma(h))^p d\mu(h)\right)^{1/p} < \infty\},$$

and for $p = \infty$ by

$$L_{\infty,w}(X) := \{f \text{ measurable on } X : \|f\|_{L_{\infty,w}} := \operatorname*{ess\,sup}_{h\in X} |f(h)|w(\sigma(h)) < \infty\}.$$

Throughout this paper, we impose the fundamental condition

$$\int_X |R(h,l)| \frac{w(\sigma(h))}{w(\sigma(l))} \, d\mu(h) \leq C_\psi \tag{3.1}$$

with a constant C_ψ independent of l. This implies by (2.3) that $V_\psi(U(\sigma(l)^{-1})\psi) \in L_{1,w}(X)$ for all $l \in X$.

The first problem is to provide a suitable large set that may serve as a reservoir of selection for the objects of our coorbit spaces. By $H'_{1,w}$ we denote the space of all continuous linear functionals on the linear space

$$H_{1,w} := \{f \in \mathcal{H} : \; V_\psi f \in L_{1,w}(X)\}.$$

The norm $\|\cdot\|_{H_{1,w}}$ on $H_{1,w}$ is defined as

$$\|f\|_{H_{1,w}} := \|V_\psi f\|_{L_{1,w}}.$$

Lemma 3.1 *The following dense and continuous embeddings hold true*

$$H_{1,w} \hookrightarrow \mathcal{H} \hookrightarrow H'_{1,w}.$$

Proof: By (3.1) we observe that $\mathcal{O}_\sigma \subset H_{1,w}$. Since $\mathcal{O}_\sigma$ is total in $\mathcal{H}$ we conclude that $H_{1,w}$ is dense in $\mathcal{H}$. Further, for $f \in H_{1,w}$, we have by the Schwarz inequality that

$$\begin{aligned} \|f\|_{\mathcal{H}}^2 &= ||V_\psi f||_{L_2}^2 = \int_X |\langle f, U(\sigma(h)^{-1})\psi\rangle_{\mathcal{H}}| \, |V_\psi f(h)| \, d\mu(h) \\ &\leq ||f||_{\mathcal{H}} \, \|\psi\|_{\mathcal{H}} \int_X |V_\psi f(h)| \, d\mu(h) \\ &\leq ||f||_{\mathcal{H}} \, \|\psi\|_{\mathcal{H}} \, ||V_\psi f||_{L_{1,w}}. \end{aligned}$$

Thus, $\|f\|_{\mathcal{H}} \leq \|\psi\|_{\mathcal{H}} \, \|f\|_{H_{1,w}}$. This implies that $H_{1,w} \hookrightarrow \mathcal{H}$. The remaining part of the proof follows by Lemma A.1 presented in the appendix. ■

The operator V_ψ in (2.2) can be extended to an operator on $H'_{1,w}$ by

$$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi\rangle_{H'_{1,w} \times H_{1,w}} \, . \tag{3.2}$$

Since $\mathcal{O}_\sigma$ is total in $H_{1,w}$, the operator V_ψ is injective. For later use, we define an operator $\tilde{V}_\psi$ on $L_{\infty,\frac{1}{w}}$ by

$$\langle \tilde{V}_\psi F, g\rangle_{H'_{1,w}\times H_{1,w}} := \langle F, V_\psi g\rangle \qquad \text{for all } g \in H_{1,w}\,. \tag{3.3}$$

The properties of V_ψ and $\tilde{V}_\psi$ are explained in the following lemma.

Lemma 3.2 *The operator V_ψ defined by* (3.2) *is a bounded operator from $H'_{1,w}$ to $L_{\infty,\frac{1}{w}}$. The operator $\tilde{V}_\psi$ defined by* (3.3) *is a bounded operator from $L_{\infty,\frac{1}{w}}$ to $H'_{1,w}$. The operators V_ψ and $\tilde{V}_\psi$ satisfy*

$$V_\psi \tilde{V}_\psi F(h) = \langle F, R(h,\cdot)\rangle \tag{3.4}$$

for all $F \in L_{\infty,\frac{1}{w}}$.

Proof: For any $f \in H'_{1,w}$, we obtain by (3.1) that

$$\begin{aligned}
\|V_\psi f\|_{L_{\infty,\frac{1}{w}}} &= \|\langle f, U(\sigma(\cdot)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}}\|_{L_{\infty,\frac{1}{w}}} \\
&\leq \|f\|_{H'_{1,w}} \operatorname*{ess\,sup}_{h\in X} \frac{1}{w(\sigma(h))}\|U(\sigma(h)^{-1})\psi\|_{H_{1,w}} \\
&\leq C_\psi \|f\|_{H'_{1,w}}\ .
\end{aligned} \tag{3.5}$$

Thus, $V_\psi : H'_{1,w} \to L_{\infty,\frac{1}{w}}(X)$ is a bounded operator.

For $F \in L_{\infty,\frac{1}{w}}(X)$ we have

$$\begin{aligned}
\|\tilde{V}_\psi F\|_{H'_{1,w}} &= \sup_{\|g\|_{H_{1,w}}=1} |\langle \tilde{V}_\psi F, g\rangle_{H'_{1,w}\times H_{1,w}}| = \sup_{\|g\|_{H_{1,w}}=1} |\langle F, V_\psi g\rangle| \\
&\leq \sup_{\|g\|_{H_{1,w}}=1} \|F\|_{L_{\infty,\frac{1}{w}}} \|V_\psi g\|_{L_{1,w}} = \|F\|_{L_{\infty,\frac{1}{w}}}.
\end{aligned}$$

Consequently, $\tilde{V}_\psi : L_{\infty,\frac{1}{w}}(X) \to H'_{1,w}$ is a bounded operator. Finally, we obtain for $F \in L_{\infty,\frac{1}{w}}(X)$ that

$$\begin{aligned}
V_\psi \tilde{V}_\psi F(h) &= \langle \tilde{V}_\psi F, U(\sigma(h)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}} = \langle F, V_\psi(U(\sigma(h)^{-1})\psi)\rangle \\
&= \langle F, R(h,\cdot)\rangle\ .
\end{aligned}$$

■

Similar to the definition of coorbit spaces in [3] we define *weighted coorbit spaces* by

$$M_{p,w} := \{f \in H'_{1,w} :\ V_\psi f \in L_{p,w}(X)\},$$

with $1 \leq p \leq \infty$ and norm

$$\|f\|_{M_{p,w}} := \|V_\psi f\|_{L_{p,w}}.$$

It is straightforward that $\|\cdot\|_{M_{p,w}}$ defines a seminorm. The property that $\|f\|_{M_{p,w}} = 0$, *i.e.*, $V_\psi f = 0$, implies $f = 0$ follows by Lemma 3.1 and since $\mathcal{O}_\sigma$ is total in $\mathcal{H}$.

The weighted coorbit spaces $M_{p,w}$ are closely related to the subspaces

$$\mathcal{M}_{p,w} := \{F \in L_{p,w}(X) : \langle F, R(h,\cdot)\rangle = F(h)\}$$

of $L_{p,w}(X)$ with $1 \le p \le \infty$. More precisely, the following fundamental correspondence principle holds true:

Theorem 3.1 *Let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{P},\sigma)$ and ψ a strictly admissible function. Assume that the kernel R fulfills* (3.1).

i) For every $f \in M_{p,w}$, the following equation is satisfied

$$\langle V_\psi f, R(h,\cdot)\rangle = V_\psi f(h)\,,$$

i.e., $V_\psi f \in \mathcal{M}_{p,w}$.

ii) For every $F \in \mathcal{M}_{p,w}$, $1 \le p \le \infty$, there exists a uniquely determined functional $f \in M_{p,w}$ such that $F = V_\psi f$.

Consequently, the spaces $M_{p,w}$ and $\mathcal{M}_{p,w}$, $1 \le p \le \infty$, are isometrically isomorphic.

Proof: *i)* Since $U(\sigma(h)^{-1})\psi \in \mathcal{H}$ we have by (2.5) that

$$\begin{aligned} V_\psi f(h) &= \langle f, U(\sigma(h)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}} \\ &= \langle f, \int_X R(h,l)U(\sigma(l)^{-1})\psi\, d\mu(l)\rangle_{H'_{1,w}\times H_{1,w}}. \end{aligned}$$

For $f \in \mathcal{H}$, we obtain by (3.1) that

$$\int_X |R(h,l)|\,|\langle f, U(\sigma(l)^{-1})\psi\rangle_{\mathcal{H}}|\, d\mu(l) \le C_\psi\, w(\sigma(h))\, \|f\|_{\mathcal{H}}\, \|\psi\|_{\mathcal{H}}.$$

Thus, by the Fubini theorem, one may change the order of integration. By Lemma 3.1 this remains true for $f \in H'_{1,w}$. We obtain

$$\begin{aligned} V_\psi f(h) &= \int_X \overline{R(h,l)}\langle f, U(\sigma(l)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}}\, d\mu(l) \\ &= \langle V_\psi f, R(h,\cdot)\rangle\ . \end{aligned}$$

ii). For $F \in \mathcal{M}_{p,w}$, $1 \le p \le \infty$, we have that

$$\begin{aligned} \|F\|_{L_{\infty,\frac{1}{w}}} &= \|\int_X F(l)\overline{R(\cdot,l)}\, d\mu(l)\|_{L_{\infty,\frac{1}{w}}} \\ &= \operatorname*{ess\,sup}_{h\in X} |\int_X F(l)\overline{R(h,l)}\, d\mu(l)|\, \frac{1}{w(\sigma(h))}. \end{aligned}$$

By applying the Hölder inequality with $1/p + 1/q = 1$ and using $w(g) \geq 1$ for all $g \in \mathcal{G}$ we further conclude that

$$
\begin{aligned}
|\int_X F(l)\overline{R(h,l)}\, d\mu(l)| &\leq \int_X |F(l)|\, w(\sigma(l))\, |R(h,l)|\, d\mu(l) \\
&\leq \left(\int_X |F(l)|^p\, w^p(\sigma(l))\, |R(h,l)|\, d\mu(l) \right)^{1/p} \times \\
&\quad \left(\int_X |R(h,l)|\, d\mu(l) \right)^{1/q} .
\end{aligned}
$$

Therefore, by (2.4), (3.1) and since $w(g) \geq 1$ for all $g \in \mathcal{G}$, we obtain

$$
\|F\|_{L_{\infty,\frac{1}{w}}} \leq C_\psi^{1/q}\, \|\psi\|_{\mathcal{H}}^{2/p}\, \|F\|_{L_{p,w}} .
$$

Thus, $F \in L_{\infty,\frac{1}{w}}(X)$ and by (3.4) we obtain that $F = V_\psi(\tilde{V}_\psi F)$, where $\tilde{V}_\psi F \in H'_{1,w}$ and since $F \in L_{p,w}(X)$ also $\tilde{V}_\psi F \in M_{p,w}$. The uniqueness condition follows by definition of $M_{p,w}$. ■

In a similar way we can prove a correspondence principle between the spaces $M_{p,\frac{1}{w}}$ and $\mathcal{M}_{p,\frac{1}{w}}$.

Corollary 3.1 *Let U be a strictly square integrable representation of $\mathcal{G}$* mod $(\mathcal{P}, \sigma)$ *and ψ a strictly admissible function. Assume that the kernel R fulfills* (3.1).

i) For every $f \in M_{p,\frac{1}{w}}$, the following equation is satisfied

$$
\langle V_\psi f, R(h,\cdot)\rangle = V_\psi f(h) ,
$$

i.e., $V_\psi f \in \mathcal{M}_{p,\frac{1}{w}}$.

ii) Assume that the kernel R satisfies the additional condition

$$
\sup_{h,l\in\mathcal{G}} |R(h,l)| \frac{w(\sigma(l))}{w(\sigma(h))} \leq C_\psi . \tag{3.6}
$$

Then, for every $F \in \mathcal{M}_{p,\frac{1}{w}}$, $1 \leq p \leq \infty$, there exists a uniquely determined functional $f \in M_{p,\frac{1}{w}}$ such that $F = V_\psi f$.

Proof: The proof of part *i)* is almost the same as those of Theorem 3.1.
To show *ii)*, we conclude by using (3.6) that

$$
|\int_X F(l)\overline{R(h,l)}\, d\mu(l)| \leq \int_X |F(l)| \frac{1}{w(\sigma(l))} (|R(h,l)| w(\sigma(l)))^{1/p+1/q}\, d\mu(l)
$$

$$\leq \left(\int\limits_X |F(l)|^p \frac{1}{w(\sigma(l))^p} |R(h,l)| w(\sigma(l)) d\mu(l) \right)^{1/p} \times$$
$$\left(\int\limits_X |R(h,l)| w(\sigma(l)) d\mu(l) \right)^{1/q}$$
$$\leq C_\psi \, w(\sigma(h)) \, \|F\|_{L_{p,\frac{1}{w}}} .$$

The rest of the proof can be performed by following the lines of the proof of Theorem 3.1 *ii).* ∎

Applying Corollary 3.1 *i)* and (3.4) we get for $f \in H'_{1,w}$ that

$$V_\psi \tilde{V}_\psi (V_\psi f)(h) = \langle V_\psi f, R(h,\cdot) \rangle = V_\psi f(h) \ .$$

Hence, $\tilde{V}_\psi V_\psi$ is the identity on $H'_{1,w}$ and we have the reconstruction formula

$$f = \tilde{V}_\psi V_\psi f = \int\limits_X \langle f, U(\sigma(h)^{-1})\psi \rangle_{H'_{1,w} \times H_{1,w}} U(\sigma(h)^{-1})\psi \, d\mu(h) \ .$$

Further we can establish the following relationship:

Corollary 3.2 *The spaces $M_{\infty,\frac{1}{w}}$ and $H'_{1,w}$ coincide,*

$$M_{\infty,\frac{1}{w}} = H'_{1,w}.$$

Proof: For $f \in H'_{1,w}$ we have by (3.5) that $\|V_\psi f\|_{L_{\infty,\frac{1}{w}}} \leq C_\psi ||f||_{H'_{1,w}}$. Conversely, we have for $f \in M_{\infty,\frac{1}{w}}$

$$\begin{aligned} \|f\|_{H'_{1,w}} &= \sup_{\|g\|_{H_{1,w}}=1} |\langle f, g\rangle_{H'_{1,w}\times H_{1,w}}| = \sup_{\|g\|_{H_{1,w}}=1} |\langle \tilde{V}_\psi V_\psi f, g\rangle_{H'_{1,w}\times H_{1,w}}| \\ &= \sup_{\|g\|_{H_{1,w}}=1} |\langle V_\psi f, V_\psi g\rangle| \leq \|V_\psi f\|_{L_{\infty,\frac{1}{w}}} . \end{aligned}$$

∎

The next natural question is to which extent the spaces $M_{p,w}$ are independent of the choice of the analyzing function and of the section. In the following lemma, we classify the admissible pairs which give rise to the same coorbit spaces.

Lemma 3.3 *Let (ψ, σ) and (η, τ) be two strictly admissible pairs such that the corresponding kernels $R_{(\psi,\sigma)}$ and $R_{(\eta,\tau)}$ satisfy* (3.1)*. Moreover, let us suppose that for some suitable constant* $0 < C < \infty$

$$\frac{1}{C} w(\sigma(h)) \leq w(\tau(h)) \leq C w(\sigma(h)) \qquad \textit{for all} \quad h \in X. \tag{3.7}$$

Let $\mathcal{R}(h,l) := \langle U(\sigma(h)^{-1})\psi, U(\tau(l)^{-1})\eta\rangle_{\mathcal{H}}$ *fulfill the conditions*

$$\int_X |\mathcal{R}(h,l)| \frac{w(\sigma(h))}{w(\sigma(l))}\, d\mu(h) \;\le\; C_{\mathcal{R}}\,, \quad \int_X |\mathcal{R}(h,l)| \frac{w(\sigma(h))}{w(\sigma(l))}\, d\mu(l) \;\le\; C_{\mathcal{R}}, \tag{3.8}$$

$$\int_X |\mathcal{R}(h,l)| \frac{w(\sigma(l))}{w(\sigma(h))}\, d\mu(h) \;\le\; C_{\mathcal{R}}\,, \quad \int_X |\mathcal{R}(h,l)| \frac{w(\sigma(l))}{w(\sigma(h))}\, d\mu(l) \;\le\; C_{\mathcal{R}} \tag{3.9}$$

with a constant $0 < C_{\mathcal{R}} < \infty$ *independent of* h *and* l*, respectively. Then the norms* $\|f\|_{M_{p,w,\psi,\sigma}}$ *and* $\|f\|_{M_{p,w,\eta,\tau}}$ *are equivalent.*

Proof: By (3.7), the norms $\|\cdot\|_{L_{p,w,\sigma}}$ and $\|\cdot\|_{L_{p,w,\tau}}$ are equivalent, so that we drop the third index as before.

Next, we show that the spaces $H_{1,w,\psi,\sigma}$ and $H_{1,w,\eta,\tau}$ have equivalent norms, which implies also the equivalence of the corresponding dual spaces. Let $f \in H_{1,w,\psi,\sigma}$. By (2.1) and definition of V_ψ and V_η, we have that

$$\int_X V_\psi f(h)\mathcal{R}(h,l)\, d\mu(h) = V_\eta f(l)$$

and consequently

$$\begin{aligned}
\|f\|_{H_{1,w,\eta,\tau}} &= \int_X |V_\eta f(l)|\, w(\tau(l))\, d\mu(l) \\
&\le \int_X \int_X |V_\psi f(h)||\mathcal{R}(h,l)|\, d\mu(h)\, w(\tau(l))\, d\mu(l) \\
&= \int_X |V_\psi f(h)|\, w(\sigma(h)) \int_X |\mathcal{R}(h,l)| \frac{w(\tau(l))}{w(\sigma(h))}\, d\mu(l)\, d\mu(h).
\end{aligned}$$

Using (3.9) and (3.7), we obtain that

$$\|f\|_{H_{1,w,\eta,\tau}} \le C\, \|V_\psi f\|_{L_{1,w}} = C\, \|f\|_{H_{1,w,\psi,\sigma}}\,.$$

By changing the roles of (ψ,σ) and (η,τ) and using (3.8) instead of (3.9), we can also prove the opposite inequality. In the following, we write again $H_{1,w}$ instead of $H_{1,w,\psi,\sigma}$ or $H_{1,w,\eta,\tau}$.

Now, for $F \in L_{p,w}$ we obtain

$$\begin{aligned}
V_\psi \tilde{V}_\eta F(h) &= \langle \tilde{V}_\eta F, U(\sigma(h)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}} \;=\; \langle F, V_\eta U(\sigma(h)^{-1})\psi\rangle \\
&= \langle F, \mathcal{R}(h,\cdot)\rangle.
\end{aligned}$$

Therefore, by applying (3.8) and the generalized Young inequality as discussed in the appendix, we get

$$\|V_\psi \tilde{V}_\eta F\|_{L_{p,w}} \le C\|F\|_{L_{p,w}}. \tag{3.10}$$

Setting $F := V_\eta f$ with $f \in M_{p,w,\eta,\tau}$ and regarding that $\tilde{V}_\eta V_\eta$ is the identity on $H'_{1,w}$, we conclude from (3.10) that

$$\|f\|_{M_{p,w,\psi,\sigma}} = \|V_\psi f\|_{L_{p,w}} \leq C\|V_\eta f\|_{L_{p,w}} = C\|f\|_{M_{p,w,\eta,\tau}}.$$

By interchanging the roles of ψ and η and using (3.9) the assertion follows. ■

4 Atomic Decompositions and Banach Frames for Weighted Coorbit Spaces

In this section, we derive some atomic decompositions for the weighted coorbit spaces established above and construct suitable Banach frames. The results are essentially a generalization of [3]. Consequently, for comparability reasons, our presentation follows the lines of [3]: Subsection 4.1 states, after some preparations, our main theorems. These results are proved by analyzing some suitable approximation operators in Subsection 4.2 and by verifying a couple of lemmas concerning frame bounds in Subsection 4.3. Here we were again inspired by the pioneering work of Feichtinger and Gröchenig, [10, 11, 12, 13]. Finally, in Subsection 4.4, we also briefly discuss atomic decompositions of the dual spaces.

4.1 Setting and Main Results

Before we can state and prove our main results, some preparations are necessary. Given some compact neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$, a family $\mathcal{X} = (x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ is called *$\mathcal{U}$–dense* if $\bigcup_{i\in\mathcal{I}} \mathcal{U}x_i = \mathcal{G}$. A family $\mathcal{X} = (x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ is called *relatively separated,* if for some compact neighborhood $\mathcal{Q}$ of the identity there exists a finite partition of the index set $\mathcal{I}$, i.e., $\mathcal{I} = \bigcup_{r=1}^{r_0} \mathcal{I}_r$, such that $\mathcal{Q}x_i \cap \mathcal{Q}x_j = \emptyset$ for all $i, j \in \mathcal{I}_r$ with $i \neq j$. By [11], this is equivalent to the property that for any compact set $\mathcal{K} \subset \mathcal{G}$ there exists a finite partition of $\mathcal{I}$ such that $\mathcal{K}x_i \cap \mathcal{K}x_j = \emptyset$ for $i, j, i \neq j$, in the same part of the partition. Note that these technical conditions can be easily fulfilled by some families $\mathcal{X}$ in all the settings we are interested in.

Let $\mathcal{U}$ be an arbitrary neighborhood of the identity in $\mathcal{G}$. By [8], there exists a bounded uniform partition of unity (of size $\mathcal{U}$), i.e., a family of continuous functions $(\varphi_i)_{i\in\mathcal{I}}$ on $\mathcal{G}$ such that

- $0 \leq \varphi_i(g) \leq 1$ for all $g \in \mathcal{G}$;
- there is an $\mathcal{U}$–dense, relatively separated family $(x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ such that supp $\varphi_i \subseteq \mathcal{U}x_i$;
- $\sum_{i\in\mathcal{I}} \varphi_i(g) \equiv 1$ for all $g \in \mathcal{G}$.

It can be shown that $\mathcal{X}$ can always be chosen such that $\sigma(X) \cap \mathcal{U}x_i \neq \emptyset$ implies $x_i \in \sigma(X)$, see [15]. Let

$$\mathcal{I}_\sigma := \{i \in \mathcal{I} : \ \sigma(X) \cap \mathcal{U}x_i \neq \emptyset\} \ .$$

Then there exist $h_i \in X$ such that $x_i = \sigma(h_i)$, where $i \in \mathcal{I}_\sigma$. Note that

$$\sum_{i \in \mathcal{I}_\sigma} \varphi_i(\sigma(h)) = 1 \ ,$$

where $h \in X$.

We define the $\mathcal{U}$*–oscillation* with respect to the analyzing wavelet ψ as

$$\operatorname{osc}_{\mathcal{U}}(l,h) \ := \ \sup_{u \in \mathcal{U}} |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(u^{-1}\sigma(l)\sigma(h)^{-1})\psi\rangle_{\mathcal{H}}|.$$

For later use, let us also remark that

$$\operatorname{osc}_{\mathcal{U}}(h,l) \ = \ \sup_{u \in \mathcal{U}} |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(\sigma(l)\sigma(h)^{-1}u)\psi\rangle_{\mathcal{H}}|.$$

In this setting, we can formulate our main theorems which we shall prove in the following subsections. The first one is a decomposition theorem which says that discretizing the representation $U(\sigma(\cdot)^{-1})$ by means of a $\mathcal{U}$–dense set indeed produces an atomic decomposition of $M_{p,w}$.

Theorem 4.1 *Let $\mathcal{G}$ be a locally compact, topological Hausdorff group with closed subgroup $\mathcal{P}$ such that $X = \mathcal{G}/\mathcal{P}$ carries an invariant measure μ. Let w be a weight function on $\mathcal{G}$. Further, let U be a strictly square integrable representation of $\mathcal{G}$* mod $(\mathcal{P},\sigma)$ *in $\mathcal{H}$ with strictly admissible function ψ. Assume that the kernel R fulfills* (3.1)*. Let a compact neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$ be chosen such that*

$$\int_X \operatorname{osc}_{\mathcal{U}}(l,h)\frac{w(\sigma(h))}{w(\sigma(l))}d\mu(l) \le \gamma \quad \text{and} \quad \int_X \operatorname{osc}_{\mathcal{U}}(l,h)\frac{w(\sigma(h))}{w(\sigma(l))}d\mu(h) \le \gamma\,, \tag{4.1}$$

where $\gamma < 1$. Let $\mathcal{X} = (x_i)_{i\in\mathcal{I}}$ be a $\mathcal{U}$–dense, relatively separated family. Furthermore, suppose that for some compact neighborhood $\mathcal{Q} \subseteq \mathcal{U}$ of the identity

$$\mu\{h \in X : \sigma(h) \in \mathcal{Q}\sigma(h_i)\} \ge C_{\mathcal{Q}} > 0 \tag{4.2}$$

holds for all $i \in \mathcal{I}_\sigma$ and that

$$\int_X \sup_{q \in \mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle| \frac{w(\sigma(h))}{w(q^{-1}\sigma(l))}\, d\mu(l) \le \tilde{C}_{\mathcal{Q}} \tag{4.3}$$

with a constant $\tilde{C}_{\mathcal{Q}} < \infty$ independent of $h \in X$. Then $M_{p,w}$, $1 \le p \le \infty$, has the following atomic decomposition: if $f \in M_{p,w}$, $1 \le p \le \infty$, then f can be represented as

$$f = \sum_{i \in \mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi,$$

where the sequence of coefficients $(c_i)_{i\in\mathcal{I}_\sigma} = (c_i(f))_{i\in\mathcal{I}_\sigma} \in \ell_{p,w}$ *depends linearly on* f *and satisfies*

$$||(c_i)_{i\in\mathcal{I}_\sigma}||_{\ell_{p,w}} \le A||f||_{M_{p,w}}. \tag{4.4}$$

If $(c_i)_{i\in\mathcal{I}_\sigma} \in \ell_{p,w}$*, then* $f = \sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi$ *is contained in* $M_{p,w}$ *and*

$$||f||_{M_{p,w}} \le B||(c_i)_{i\in\mathcal{I}_\sigma}||_{\ell_{p,w}}. \tag{4.5}$$

Here we use $w = (w(x_i))_{i\in\mathcal{I}_\sigma}$ as discretized weight sequence and

$$\ell_{p,w} := \{c = (c_i)_{i\in\mathcal{I}_\sigma} : \|c\|_{\ell_{p,w}} := \|c\,w\|_{\ell_p} < \infty\}$$

for $1 \le p \le \infty$.

Given such an atomic decomposition, the problem arises under which conditions a function f is completely determined by the moments or coefficients $\langle f, U(\sigma(h_i)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}}$ and how f can be reconstructed from these coefficients. This question is answered by the following theorem which shows that our generalized coherent states indeed give rise to Banach frames.

Theorem 4.2 *Impose the same assumptions as in Theorem 4.1 with*

$$\int_X \mathrm{osc}_{\mathcal{U}}(h,l)\frac{w(\sigma(h))}{w(\sigma(l))}d\mu(l) \le \frac{\tilde\gamma}{C_\psi} \quad \text{and} \quad \int_X \mathrm{osc}_{\mathcal{U}}(h,l)\frac{w(\sigma(h))}{w(\sigma(l))}d\mu(h) \le \frac{\tilde\gamma}{C_\psi}, \tag{4.6}$$

where $\tilde\gamma < 1$*, instead of* (4.1) *and with*

$$\int_X \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle| \frac{w(q^{-1}\sigma(l))}{w(\sigma(h))}\, d\mu(l) \le \tilde C_{\mathcal{Q}} \tag{4.7}$$

where $\tilde C_{\mathcal{Q}} < \infty$ *is a constant independent of* $h \in X$*, instead of* (4.3)*. Let* R *fulfill the additional property*

$$\int_X |R(h,l)|\frac{w(\sigma(h))}{w(\sigma(l))}\, d\mu(l) \le C_\psi. \tag{4.8}$$

Then the set

$$\{\psi_i := U(\sigma(h_i)^{-1})\psi : i \in \mathcal{I}_\sigma\}$$

is a Banach frame for $M_{p,w}$*. This means that*

i) $f \in M_{p,w}$ *if and only if* $(\langle f, \psi_i\rangle_{H'_{1,w}\times H_{1,w}})_{i\in\mathcal{I}_\sigma} \in \ell_{p,w}$*;*

ii) there exist two constants $0 < A' \le B' < \infty$ *such that*

$$A'\,\|f\|_{M_{p,w}} \le \|(\langle f, \psi_i\rangle_{H'_{1,w}\times H_{1,w}})_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,w}} \le B'\,\|f\|_{M_{p,w}}; \tag{4.9}$$

iii) there exists a bounded, linear reconstruction operator $\mathcal{S}$ *from* $\ell_{p,w}$ *to* $M_{p,w}$ *such that* $\mathcal{S}\left((\langle f, \psi_i\rangle_{H'_{1,w}\times H_{1,w}})_{i\in\mathcal{I}_\sigma}\right) = f$.

For further information concerning Banach frames, we refer to [16].

4.2 Approximation Operators

In this section, we examine two different approximation operators on $\mathcal{M}_{p,w}$. We use the results to construct expansions for the spaces $\mathcal{M}_{p,w}$, which then, by the correspondence principle in Theorem 3.1, lead to expansions for the coorbit spaces $M_{p,w}$.

We consider the following approximation operators:

$$\begin{aligned}
T_\varphi F(h) &:= \sum_{i\in\mathcal{I}_\sigma} \langle F, \varphi_i \circ \sigma\rangle \, R(h_i, h) \\
&= \sum_{i\in\mathcal{I}_\sigma} \int_X F(l)\varphi_i(\sigma(l))\, d\mu(l) R(h_i, h), \\
S_\varphi F(h) &:= \sum_{i\in\mathcal{I}_\sigma} F(h_i)\langle \varphi_i \circ \sigma, R(h,\cdot)\rangle \\
&= \sum_{i\in\mathcal{I}_\sigma} \int_X F(h_i)\varphi_i(\sigma(l))R(l,h)\, d\mu(l).
\end{aligned}$$

So far, it is not clear a prior whether these formal expressions make sense at all and on which spaces they are bounded operators. This will be clarified in Theorem 4.3 below. Another remark is required on the meaning of the sum over $\mathcal{I}_\sigma$. We order the finite subsets of $\mathcal{I}_\sigma$ by inclusion, then $\sum_{i\in\mathcal{I}_\sigma} \ldots$ will be understood as the limit of the partial sums over finite subsets of $\mathcal{I}_\sigma$.

The first step is to establish the invertibility of the operators T_φ and S_φ.

Theorem 4.3 *i) If the conditions* (4.1) *are fulfilled, then the operator* $T_\varphi : \mathcal{M}_{p,w} \to \mathcal{M}_{p,w}$ *is bounded with bounded inverse.*
ii) If the conditions (4.8) *and* (4.6) *are fulfilled, then the operator* $S_\varphi : \mathcal{M}_{p,w} \to \mathcal{M}_{p,w}$ *is bounded with bounded inverse.*

Proof: By definition of $\mathcal{M}_{p,w}$, we have for $F \in \mathcal{M}_{p,w}$ that

$$\begin{aligned}
F(h) &= \langle F, R(h,\cdot)\rangle = \int_X F(l)\overline{R(h,l)}\, d\mu(l) \\
&= \sum_{i\in\mathcal{I}_\sigma} \int_X F(l)\varphi_i(\sigma(l))R(l,h)\, d\mu(l)
\end{aligned}$$

and consequently

$$\begin{aligned}
F(h) - T_\varphi F(h) &= \sum_{i\in\mathcal{I}_\sigma} \int_X F(l)\varphi_i(\sigma(l))\, [R(l,h) - R(h_i,h)]\, d\mu(l), \\
F(h) - S_\varphi F(h) &= \sum_{i\in\mathcal{I}_\sigma} \int_X [F(l) - F(h_i)]\varphi_i(\sigma(l))R(l,h)\, d\mu(l). \qquad (4.10)
\end{aligned}$$

Let us first consider $F - T_\varphi F$. By the definition of R we obtain

$$\begin{aligned}|F(h) - T_\varphi F(h)| &\leq \sum_{i\in\mathcal{I}_\sigma} \int_X |F(l)|\varphi_i(\sigma(l))|R(l,h) - R(h_i,h)|\, d\mu(l)\\ &= \sum_{i\in\mathcal{I}_\sigma} \int_X |F(l)|\varphi_i(\sigma(l)) \times\\ &\quad |\langle\psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(\sigma(h_i)\sigma(h)^{-1})\psi\rangle_{\mathcal{H}}|\, d\mu(l).\end{aligned}$$

Now $\sigma(l) \in \mathcal{U}x_i$ implies that there exists $u \in \mathcal{U}$ such that $\sigma(l) = u\, x_i = u\, \sigma(h_i)$. Thus $\sigma(h_i) = u^{-1}\sigma(l)$ and we get

$$|F(h) - T_\varphi F(h)| \leq \sum_{i\in\mathcal{I}_\sigma} \int_X |F(l)|\varphi_i(\sigma(l))\, \mathrm{osc}_{\mathcal{U}}(l,h)\, d\mu(l) = \int_X |F(l)|\, \mathrm{osc}_{\mathcal{U}}(l,h)\, d\mu(l).$$

By recalling the assumptions (4.1) and applying the weighted Young inequality (see appendix), we obtain

$$\|F - T_\varphi F\|_{L_{p,w}} = \|(I - T_\varphi)F\|_{L_{p,w}} \leq \gamma\, \|F\|_{L_{p,w}}.$$

Consequently $|||I - T_\varphi||| < 1$, i.e., $I - T_\varphi$ is a contraction on $\mathcal{M}_{p,w}$. Thus, regarding that $|||T_\varphi||| \leq |||T_\varphi - I||| + |||I|||$, we see that T_φ is a bounded operator with bounded inverse.

Next we consider $F - S_\varphi F$. Since $F \in \mathcal{M}_{p,w}$ and by the definition of R we obtain

$$\begin{aligned}|F(l) - F(h_i)| &\leq \int_X |F(g)||R(g,l) - R(g,h_i)|\, d\mu(g)\\ &= \int_X |F(g)||\langle\psi, U(\sigma(g)\sigma(l)^{-1})\psi - U(\sigma(g)\sigma(h_i)^{-1})\psi\rangle_{\mathcal{H}}|\, d\mu(g).\end{aligned}$$

By (4.10) we are only interested in $l \in X$ with $\sigma(l) \in \mathcal{U}x_i$, i.e., $\sigma(l) = u\, \sigma(h_i)$ for some $u \in \mathcal{U}$ and hence $\sigma(h_i)^{-1} = \sigma(l)^{-1}u$. Thus

$$|F(l) - F(h_i)| \leq \int_X |F(g)|\, \mathrm{osc}_{\mathcal{U}}(l,g)\, d\mu(g)$$

and since (φ_i) is a partition of unity

$$\sum_{i\in\mathcal{I}_\sigma} |F(l) - F(h_i)|\varphi_i(\sigma(l)) \leq \int_X |F(g)|\, \mathrm{osc}_{\mathcal{U}}(l,g)\, d\mu(g).$$

Then we obtain by (3.1), (4.8), (4.10) and the weighted Young inequality

$$\|F - S_\varphi F\|_{L_{p,w}} \leq C_\psi\, ||\sum_{i\in\mathcal{I}_\sigma} |F(\cdot) - F(h_i)|\varphi_i(\sigma(\cdot))||_{L_{p,w}} \leq C_\psi \frac{\tilde{\gamma}}{C_\psi}\, ||F||_{L_{p,w}}.$$

Consequently, $I - S_\varphi$ is a contraction on $\mathcal{M}_{p,w}$ and S_φ is a bounded operator with bounded inverse on $\mathcal{M}_{p,w}$. ■

Using the correspondence principle we can derive the following representation of functions from our coorbit spaces.

Corollary 4.1 *Any function $f \in M_{p,w}$ can be decomposed as*

$$f = \sum_{i\in\mathcal{I}_\sigma} c_i\, U(\sigma(h_i)^{-1})\psi \ , \tag{4.11}$$

where

$$c_i = c_i(f) := \langle T_\varphi^{-1} F, \varphi_i \circ \sigma\rangle$$

and $F := V_\psi f$.

Proof: By Theorem 3.1 *i)* and Theorem 4.3 *i)* we have that

$$V_\psi f(h) = F(h) = T_\varphi T_\varphi^{-1} F(h) = \sum_{i\in\mathcal{I}_\sigma} \langle T_\varphi^{-1} F, \varphi_i \circ \sigma\rangle R(h_i, h).$$

Since $\tilde{V}_\psi V_\psi$ is the identity on $H'_{1,w}$ and $\tilde{V}_\psi$ is bounded on $L_{\infty,\frac{1}{w}}$, we obtain

$$f = \tilde{V}_\psi V_\psi f = \sum_{i\in\mathcal{I}_\sigma} \langle T_\varphi^{-1} F, \varphi_i \circ \sigma\rangle \tilde{V}_\psi(R(h_i,\cdot)). \tag{4.12}$$

Now, for any $g \in H_{1,w}$,

$$\begin{aligned} \langle \tilde{V}_\psi(R(h_i,\cdot)), g\rangle_{H'_{1,w}\times H_{1,w}} &= \langle R(h_i,\cdot), V_\psi g\rangle \ = \ \overline{V_\psi g(h_i)} \\ &= \langle U(\sigma(h_i)^{-1})\psi, g\rangle_{H'_{1,w}\times H_{1,w}} \end{aligned}$$

so that $\tilde{V}_\psi(R(h_i,\cdot)) = U(\sigma(h_i)^{-1})\psi$. Together with (4.12) this yields the assertion. ■

Moreover, the operator S_φ induces the reconstruction operator as stated in Theorem 4.2 *iii)*.

Corollary 4.2 *Any function $f \in M_{p,w}$ can be reconstructed as*

$$f = \sum_{i\in\mathcal{I}_\sigma} \langle f, U(\sigma(h_i)^{-1}\psi\rangle_{H'_{1,w}\times H_{1,w}} e_i,$$

where

$$e_i = \tilde{V}_\psi(E_i), \quad E_i(h) := S_\varphi^{-1}(\langle \varphi_i \circ \sigma, R(h,\cdot)\rangle).$$

Proof: Since S_φ has a continuous inverse, we obtain for $F := V_\psi f \in \mathcal{M}_{p,w}$ that

$$\begin{aligned} F(h) &= S_\varphi^{-1} S_\varphi F(h) \\ &= \sum_{i\in\mathcal{I}_\sigma} F(h_i) S_\varphi^{-1}\langle \varphi_i \circ \sigma, R(h,\cdot)\rangle \ = \ \sum_{i\in\mathcal{I}_\sigma} F(h_i) E_i(h). \end{aligned}$$

Now the correspondence principle and the continuity of $\tilde{V}_\psi$ on $L_{\infty,\frac{1}{w}}$ implies

$$
\begin{aligned}
f &= \tilde{V}_\psi V_\psi f = \tilde{V}_\psi \left(\sum_{i\in\mathcal{I}_\sigma} V_\psi(f)(h_i) E_i \right) \\
&= \sum_{i\in\mathcal{I}_\sigma} \langle f, U(\sigma(h_i)^{-1}\psi\rangle_{H'_{1,w}\times H_{1,w}} \tilde{V}_\psi(E_i) = \sum_{i\in\mathcal{I}_\sigma} \langle f, U(\sigma(h_i)^{-1}\psi\rangle_{H'_{1,w}\times H_{1,w}} e_i .
\end{aligned}
$$

■

4.3 Frame Bounds

In this section, we prove the norm equivalences in Theorem 4.1 and 4.2. For the verification that the infinite sums appearing in the following lemmatas converge (unconditionally) in $\mathcal{M}_{p,w}$, respectively $M_{p,w}$, it suffices to obtain for $p < \infty$ the estimates for finite sequences. Then all the estimates can be extended in the usual way, see again [10, 11, 12] for details. Only the case $p = \infty$ requires some additional effort. The necessary modifications are left to the reader.

In the following, 'C' always denotes a generic constant which is independent of all the other parameters under consideration, but whose concrete value may be different in each particular estimate.

We start with Theorem 4.1, relation (4.4).

Lemma 4.1 *Suppose that the conditions in Theorem 4.1 are satisfied. For any $f \in M_{p,w}$ let*

$$(c_i)_{i\in\mathcal{I}_\sigma} := (\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle)_{i\in\mathcal{I}_\sigma}.$$

Then there exists a constant $A < \infty$ such that the following inequality holds:

$$\|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,w}} \le A\|f\|_{M_{p,w}} .$$

In particular, we have that $(c_i)_{i\in\mathcal{I}_\sigma} \in \ell_{p,w}$.

Proof: 1. First of all, we show that for any sequence $(\eta_i)_{i\in\mathcal{I}_\sigma}$ the inequality

$$\|(\eta_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,w}} \le C \,\|\sum_{i\in\mathcal{I}_\sigma} |\eta_i| 1_{\mathcal{U}x_i} \circ \sigma\|_{L_{p,w}} \tag{4.13}$$

holds, where again $x_i = \sigma(h_i)$ and where $1_{\mathcal{U}x_i}$ denotes the characteristic function of $\mathcal{U}x_i$.

Since $(x_i)_{i\in\mathcal{I}}$ is a relatively separated family, there exists a splitting $\mathcal{I} = \bigcup_{r=1}^{r_0} \mathcal{I}_r$ such that $\mathcal{U}x_i \cap \mathcal{U}x_j = \emptyset$ for $i, j \in \mathcal{I}_r$ and $i \neq j$. This results in a decomposition $\mathcal{I}_\sigma = \bigcup_{r=1}^{r_0} \mathcal{I}_{\sigma r}$, where

$$\mathcal{I}_{\sigma r} = \{i \in \mathcal{I}_r : \ \mathcal{U}x_i \cap \sigma(X) \neq \emptyset\} .$$

Then we obtain

$$\begin{aligned}\|\sum_{i\in\mathcal{I}_\sigma}|\eta_i|1_{\mathcal{U}x_i}\circ\sigma\|^p_{L_{p,w}} &= \int_X\left(\sum_{r=1}^{r_0}\sum_{i\in\mathcal{I}_{\sigma r}}|\eta_i|1_{\mathcal{U}x_i}(\sigma(h))\,w(\sigma(h))\right)^p d\mu(h)\\ &\geq \sum_{r=1}^{r_0}\int_X\left(\sum_{i\in\mathcal{I}_{\sigma r}}|\eta_i|1_{\mathcal{U}x_i}(\sigma(h))\,w(\sigma(h))\right)^p d\mu(h)\\ &= \sum_{r=1}^{r_0}\int_X\sum_{i\in\mathcal{I}_{\sigma r}}|\eta_i|^p1_{\mathcal{U}x_i}(\sigma(h))\,w^p(\sigma(h))d\mu(h).\end{aligned}$$

Moreover, since $w(x_i)\leq w(u^{-1})w(\sigma(h))$ for $\sigma(h)\in\mathcal{U}x_i$ and $\mathcal{Q}\subset\mathcal{U}$, we can conclude from (4.2) that

$$\|\sum_{i\in\mathcal{I}_\sigma}|\eta_i|1_{\mathcal{U}x_i}\circ\sigma\|^p_{L_{p,w}} \;\geq\; (\max_{u\in\mathcal{U}}w(u^{-1}))^{-p}\,C_{\mathcal{Q}}\sum_{i\in\mathcal{I}_\sigma}|\eta_i|^p\,w^p(x_i)$$

which implies (4.13) by continuity of w and since $\mathcal{U}$ is compact.
2. Let $F\in L_{p,w}$. Then the application of (4.13) yields

$$\begin{aligned}\|(\langle F,\varphi_i\circ\sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,w}} &\leq \|(\langle|F|,\varphi_i\circ\sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,w}}\\ &\leq C\,\|\sum_{i\in\mathcal{I}_\sigma}\langle|F|,\varphi_i\circ\sigma\rangle1_{\mathcal{U}x_i}\circ\sigma\|_{L_{p,w}}\ .\end{aligned}$$

Further, we see for an arbitrary fixed $h\in X$ that

$$\sum_{i\in\mathcal{I}_\sigma}\langle|F|,\varphi_i\circ\sigma\rangle1_{\mathcal{U}x_i}(\sigma(h))=\sum_{i\in\mathcal{I}_h}\langle|F|,\varphi_i\circ\sigma\rangle\ ,$$

where $\mathcal{I}_h:=\{i\in\mathcal{I}_\sigma:\ \sigma(h)\in\mathcal{U}x_i\}$. Since $(x_i)_{i\in\mathcal{I}}$ is a relatively separated family, we see, by using the notation in the first part of the proof, that $\#\mathcal{I}_h\leq r_0$ and consequently

$$\sum_{i\in\mathcal{I}_h}\langle|F|,\varphi_i\circ\sigma\rangle\leq\langle|F|,K(h,\cdot)\rangle$$

with

$$K(h,l):=\sum_{i\in\mathcal{I}_h}1_{\mathcal{U}x_i}(\sigma(l))=\sum_{i\in\mathcal{I}_l}1_{\mathcal{U}x_i}(\sigma(h)).$$

We may assume that there exists some constant C such that $\mu(\Pi(\mathcal{U}))\leq C$. For $x\in\mathcal{G}$, we have by the $\mathcal{G}$–invariance of μ and since $\Pi(\mathcal{U})x=\Pi(\mathcal{U}x)$ that

$$\mu(\Pi(\mathcal{U}x))=\mu(\Pi(\mathcal{U})x)=\mu(\Pi(\mathcal{U}))\leq C.$$

Now $\{l\in X:\sigma(l)\in\mathcal{U}x\}\subseteq\Pi(\mathcal{U}x)$ so that

$$\mu(\{l\in X:\sigma(l)\in\mathcal{U}x\})\leq\mu(\Pi(\mathcal{U}x))\leq C. \tag{4.14}$$

Further, $\sigma(h) \in \mathcal{U}x_i$ and $\sigma(l) \in \mathcal{U}x_i$ imply that $\sigma(h) = u_2 u_1^{-1}\sigma(l)$ for some $u_1, u_2 \in \mathcal{U}$ and consequently by the submultiplicativity of our weight function

$$w(\sigma(h)) = w(u_2 u_1^{-1}\sigma(l)) \leq w(u_2 u_1^{-1})\, w(\sigma(l)).$$

Since $\mathcal{U}\mathcal{U}^{-1}$ is compact and w is continuous, we obtain that $w(\sigma(h))/w(\sigma(l)) \leq C_w$ with a constant C_w independent of h and l. Together with (4.14) we conclude that

$$\int_X K(h,l)\frac{w(\sigma(h))}{w(\sigma(l))}\, d\mu(l) \leq C_w\, C\, r_0$$

for all $h \in X$ and similarly for the integration with respect to $d\mu(h)$ for all $l \in X$. Therefore the weighted Young inequality implies that

$$\|(\langle F, \varphi_i \circ \sigma\rangle)_{i\in I_\sigma}\|_{\ell_{p,w}} \leq C\, \|\langle |F|, K(h,\cdot)\rangle\|_{L_{p,w}} \leq C\, \|F\|_{L_{p,w}}\ .$$

3. Finally, we conclude by the correspondence principle and by using $F = T_\varphi^{-1} V_\psi f \in \mathcal{M}_{p,w}$ in the above inequality that

$$\begin{aligned}
\|(\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,w}} &\leq C\, \|T_\varphi^{-1} V_\psi f\|_{L_{p,w}} \\
&\leq C\, |||T_\varphi^{-1}|||\ \|V_\psi f\|_{L_{p,w}} \\
&\leq C\, |||T_\varphi^{-1}|||\ \|f\|_{M_{p,w}}\ .
\end{aligned}$$

■

The next step is to establish (4.5).

Lemma 4.2 *Suppose that the conditions in Theorem 4.1 are satisfied. Then there exists a constant $B < \infty$ such that for any sequence $(c_i)_{i\in\mathcal{I}_\sigma} \in \ell_{p,w}$, $1 \leq p \leq \infty$, the following inequality holds:*

$$\|\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi\|_{M_{p,w}} \leq B\, \|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,w}}\ .$$

Proof: 1. First we prove that

$$\|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h)\|_{L_{p,w}} \leq B\|(c_i)_{i\in I_\sigma}\|_{\ell_{p,w}}.$$

To this end, we want to use the weighted Riesz–Thorin Interpolation Theorem as outlined in the appendix. That is, we show that

$$\mathcal{T} : (c_i)_{i\in I_\sigma} \longmapsto \sum_{i\in I_\sigma} c_i R(h_i,\cdot)$$

is a bounded operator from $\ell_{1,w}$ to $L_{1,w}$ and from $\ell_{\infty,w}$ to $L_{\infty,w}$. Then the Riesz–Thorin Theorem implies that $\mathcal{T}$ is also a bounded operator from $\ell_{p,w}$ to $L_{p,w}$ for all $1 \leq p \leq \infty$.

For $p=1$, we obtain by (3.1) that

$$\begin{aligned}\|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,\cdot)\|_{L_{1,w}} &\le \int_X \sum_{i\in\mathcal{I}_\sigma} |c_i|\; |R(h_i,h)|\, w(\sigma(h))\, d\mu(h)\\ &\le \sum_{i\in\mathcal{I}_\sigma} |c_i| \int_X |R(h_i,h)| w(\sigma(h))\, d\mu(h)\\ &\le \sum_{i\in\mathcal{I}_\sigma} |c_i| C_\psi w(\sigma(h_i)) = C_\psi \|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{1,w}}\ .\end{aligned}$$

For $p=\infty$ it follows that

$$\begin{aligned}\|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,\cdot)\|_{L_{\infty,w}} &= \sup_{h\in X} |\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h)\, w(\sigma(h))\,|\\ &\le \sup_{i\in\mathcal{I}_\sigma} |c_i|\, w(\sigma(h_i)) \sup_{h\in X} \sum_{i\in\mathcal{I}_\sigma} |R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))}\\ &= \|(c_i)_{i\in I_\sigma}\|_{\ell_{\infty,w}} \sup_{h\in X} \sum_{i\in\mathcal{I}_\sigma} |R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))}. \qquad (4.15)\end{aligned}$$

Let $\mathcal{Q}$ be some compact neighborhood of the identity which satisfies (4.2) and (4.3). Since $(x_i)_{i\in\mathcal{I}}$ is a relatively separated family there exists a finite splitting $\mathcal{I}_\sigma = \bigcup_{r=1}^{r_0} \mathcal{I}_{\sigma r}$ so that $\mathcal{Q}x_i \cap \mathcal{Q}x_j = \emptyset$ for $i,j\in\mathcal{I}_{\sigma r}$ and $i\neq j$. Hence we obtain

$$\sum_{i\in\mathcal{I}_\sigma} |R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))} = \sum_{r=1}^{r_0} \sum_{i\in\mathcal{I}_{\sigma r}} |R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))}\ .$$

For all $l\in X$ with the property that $\sigma(l)\in\mathcal{Q}\sigma(h_i)$, we have that $\sigma(h_i)^{-1}\in\sigma(l)^{-1}\mathcal{Q}$ and hence

$$\begin{aligned}|R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))} &= |\langle U(\sigma(h)^{-1})\psi, U(\sigma(h_i)^{-1})\psi\rangle_{\mathcal{H}}| \frac{w(\sigma(h))}{w(\sigma(h_i))}\\ &\le \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{\mathcal{H}}|\, \frac{w(\sigma(h))}{w(q^{-1}\sigma(l))}.\end{aligned}$$

Let $\mathcal{B}_i := \{l\in X:\ \sigma(l)\in\mathcal{Q}\sigma(h_i)\}$. Then the above inequality implies

$$\int_{\mathcal{B}_i} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{\mathcal{H}}|\, \frac{w(\sigma(h))}{w(q^{-1}\sigma(l))}\, d\mu(l) \ge |R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))}\, \mu(\mathcal{B}_i)\ .$$

Now we have that the sets $\mathcal{B}_i$ and $\mathcal{B}_j$ are disjoint for $i,j\in\mathcal{I}_{\sigma r}$ and $i\neq j$. Consequently, we obtain by (4.3) and (4.2)

$$\tilde{C}_{\mathcal{Q}} \ \ge\ \int_X \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{\mathcal{H}}| \frac{w(\sigma(h))}{w(q^{-1}\sigma(l))}\, d\mu(l)\ \ge$$

$$
\begin{aligned}
&\geq \sum_{i\in\mathcal{I}_{\sigma r}} \int_{\mathcal{B}_i} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{\mathcal{H}}| \frac{w(\sigma(h))}{w(q^{-1}\sigma(l))}\, d\mu(l) \\
&\geq \sum_{i\in\mathcal{I}_{\sigma r}} |R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))}\, \mu(\mathcal{B}_i) \\
&\geq C_{\mathcal{Q}} \sum_{i\in\mathcal{I}_{\sigma r}} |R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))}
\end{aligned}
$$

and hence

$$
\sum_{i\in\mathcal{I}_{\sigma r}} |R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))} \leq \frac{\tilde{C}_{\mathcal{Q}}}{C_{\mathcal{Q}}} \quad , \quad \sum_{i\in\mathcal{I}_{\sigma}} |R(h_i,h)|\, \frac{w(\sigma(h))}{w(\sigma(h_i))} \leq \frac{r_0\tilde{C}_{\mathcal{Q}}}{C_{\mathcal{Q}}}\ . \tag{4.16}
$$

Together with (4.15) this yields

$$
\|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,\cdot)\|_{L_{\infty,w}} \leq \|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{\infty,w}} \frac{r_0\tilde{C}_{\mathcal{Q}}}{C_{\mathcal{Q}}}\ .
$$

2. Now it is easy to check that $\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h) \in \mathcal{M}_{p,w}$. Since $V_\psi\tilde{V}_\psi$ is the identity on $L_{\infty,\frac{1}{w}}$ and $\tilde{V}_\psi V_\psi$ on $H'_{1,w}$, we obtain

$$
\begin{aligned}
\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h) &= V_\psi\tilde{V}_\psi \left(\sum_{i\in\mathcal{I}_\sigma} c_i V_\psi(U(\sigma(h_i)^{-1})\psi)(h)\right) \\
&= V_\psi \left(\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi\right)(h).
\end{aligned}
$$

Thus,

$$
\|\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi\|_{M_{p,w}} = \|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,\cdot)\|_{L_{p,w}}
$$

and we are done. ■

Next let us turn to the estimates (4.9) in Theorem 4.2.

Lemma 4.3 *Suppose that the conditions in Theorem 4.2 are satisfied. For $i \in I_\sigma$, let $\psi_i := U(\sigma(h_i)^{-1})\psi$. Then, for $f \in M_{p,w}$, there exists a constant $B' < \infty$ such that*

$$
||\left(\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}}\right)_{i\in I_\sigma}||_{\ell_{p,w}} \leq B'||f||_{M_{p,w}}.
$$

Proof: Let $F := V_\psi f$. By the correspondence principle the assertion is equivalent to

$$
||\,(F(h_i))_{i\in I_\sigma}\,||_{\ell_{p,w}} \leq B'||F||_{L_{p,w}}. \tag{4.17}
$$

We prove (4.17) for $p=1$ and $p=\infty$ and apply again the weighted Riesz–Thorin Interpolation Theorem to obtain the inequality for all $1 \le p \le \infty$.

For $p=1$, we conclude as follows

$$\begin{aligned}
\sum_{i\in I_\sigma} |F(h_i)|\, w(\sigma(h_i)) &= \sum_{i\in I_\sigma} |\langle F, R(h_i,\cdot)\rangle|\, w(\sigma(h_i)) \\
&\le \sum_{i\in I_\sigma} \int_X |F(l)||R(h_i,l)|\, w(\sigma(h_i))\, d\mu(l) \\
&= \int_X |F(l)| w(\sigma(l)) \sum_{i\in I_\sigma} |R(h_i,l)| \frac{w(\sigma(h_i))}{w(\sigma(l))}\, d\mu(l) \\
&\le ||F||_{L_{1,w}} \sup_{l\in X} \sum_{i\in I_\sigma} |R(h_i,l)| \frac{w(\sigma(h_i))}{w(\sigma(l))}\,.
\end{aligned}$$

Using (4.7) we obtain as in (4.16) that $\sum_{i\in I_\sigma} |R(h_i,l)| \frac{w(\sigma(h_i))}{w(\sigma(l))} \le r_0 \tilde{C}_{\mathcal{Q}}/C_{\mathcal{Q}}$ and consequently

$$\sum_{i\in I_\sigma} |F(h_i)| w(\sigma(h_i)) \le \frac{r_0 \tilde{C}_{\mathcal{Q}}}{C_{\mathcal{Q}}}\, ||F||_{L_{1,w}}.$$

For $p=\infty$, we get

$$\begin{aligned}
\sup_{i\in I_\sigma} |F(h_i)| w(\sigma(h_i)) &= \sup_{i\in I_\sigma} |\langle F, R(h_i,\cdot)\rangle| w(\sigma(h_i)) \\
&\le \sup_{i\in I_\sigma} \int_X |F(l)||R(h_i,l)| w(\sigma(h_i))\, d\mu(l) \\
&\le \sup_{l\in X} |F(l)| w(\sigma(l)) \sup_{i\in I_\sigma} \int_X |R(h_i,l)| \frac{w(\sigma(h_i))}{w(\sigma(l))}\, d\mu(l) \\
&\le C_\psi\, ||F||_{L_{\infty,w}},
\end{aligned}$$

where we have used (4.8) for the last estimate. This finishes the proof. ■

Lemma 4.4 *Suppose that the conditions in Theorem 4.2 are satisfied. For* $i \in I_\sigma$, *let* $\psi_i := U(\sigma(h_i)^{-1})\psi$. *Then, for* $\left(\langle f, \psi_i \rangle_{H'_{1,w}\times H_{1,w}}\right)_{i\in I_\sigma} \in \ell_{p,w}$, *there exists a constant* $A' > 0$ *such that*

$$||f||_{M_{p,w}} \le \frac{1}{A'}\, || \left(\langle f, \psi_i \rangle_{H'_{1,w}\times H_{1,w}}\right)_{i\in I_\sigma} ||_{\ell_{p,w}}.$$

Proof: 1. First we show that

$$\tilde{\mathcal{T}} : (c_i)_{i\in I_\sigma} \mapsto \langle \sum_{i\in I_\sigma} c_i \varphi_i \circ \sigma, R(h,\cdot)\rangle$$

is a bounded operator from $\ell_{p,w}$ to $\mathcal{M}_{p,w}$. Again by the Riesz–Thorin Theorem, if suffices to show the boundedness for $p=1$ and $p=\infty$.

For $p=1$, we get by (3.1), (4.8) and the weighted Young inequality

$$
\begin{aligned}
&||\langle \sum_{i\in I_\sigma} c_i\,(\varphi_i\circ\sigma)(h), R(h,\cdot)\rangle||_{L_{1,w}} \;\le\; C_\psi\,||\sum_{i\in I_\sigma} c_i\,\varphi_i(\sigma(\cdot))||_{L_{1,w}}\\
&\le C_\psi \int_X \sum_{i\in I_\sigma} |c_i|\,w(\sigma(h_i))\,\varphi_i(\sigma(h))\,\frac{w(\sigma(h))}{w(\sigma(h_i))}\,d\mu(h)\\
&\le C_\psi\,||(c_i)_{i\in I_\sigma}||_{\ell_{1,w}}\,\sup_{i\in I_\sigma}\int_X |\varphi_i(\sigma(h))|\,\frac{w(\sigma(h))}{w(\sigma(h_i))}\,d\mu(h). \qquad (4.18)
\end{aligned}
$$

By $\operatorname{supp}\varphi_i\subseteq\mathcal{U}\sigma(h_i)$ we consider $h\in X$ with $\sigma(h)=u\sigma(h_i)$. Then, by using similar arguments as in the proof of Lemma 4.1, we obtain

$$
\frac{w(\sigma(h))}{w(\sigma(h_i))}\le C \qquad (4.19)
$$

with a constant C independent of h_i and h. Hence, since μ is an invariant measure, we can estimate (4.18) by

$$
||\langle \sum_{i\in I_\sigma} c_i\,\varphi_i\circ\sigma, R(h,\cdot)\rangle||_{L_{1,w}} \le C_\psi\,C\,||(c_i)_{i\in I_\sigma}||_{\ell_{1,w}}.
$$

For $p=\infty$, we obtain in a similar way by using the weighted Young inequality

$$
\begin{aligned}
||\langle \sum_{i\in I_\sigma} c_i\,\varphi_i\circ\sigma, R(h,\cdot)\rangle||_{L_{\infty,w}} &\le C_\psi\,\sup_{h\in X}|\sum_{i\in I_\sigma} c_i\,\varphi_i(\sigma(h))|\,w(\sigma(h))\\
&\le C_\psi\,\sup_{i\in I_\sigma}|c_i|\,w(\sigma(h_i))\,\sup_{h\in X}\sum_{i\in I_\sigma}\varphi_i(\sigma(h))\,\frac{w(\sigma(h))}{w(\sigma(h_i))},
\end{aligned}
$$

and further by (4.19) and since $\{\varphi_i\}$ is a partition of unity that

$$
||\langle \sum_{i\in I_\sigma} c_i\varphi_i\circ\sigma, R(h,\cdot)\rangle||_{L_{\infty,w}} \le C_\psi\,C\,||(c_i)_{i\in I_\sigma}||_{\ell_{\infty,w}}.
$$

2. Next it is easy to check that

$$
\langle \sum_{i\in I_\sigma} c_i\varphi_i\circ\sigma, R(h,\cdot)\rangle = \sum_{i\in I_\sigma} c_i\langle \varphi_i\circ\sigma, R(h,\cdot)\rangle.
$$

Since S_φ^{-1} is a bounded operator on $\mathcal{M}_{p,w}$, we conclude that

$$
(c_i)_{i\in I_\sigma}\mapsto S_\varphi^{-1}\left(\sum_{i\in I_\sigma} c_i\langle\varphi_i\circ\sigma, R(h,\cdot)\rangle\right) = \sum_{i\in I_\sigma} c_i S_\varphi^{-1}\left(\langle\varphi_i\circ\sigma, R(h,\cdot)\rangle\right)
$$

is also bounded from $\ell_{p,w}$ to $\mathcal{M}_{p,w}$.
3. Finally, we apply part 1 and 2 of the proof to the special sequence $\left(\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}}\right)_{i\in I_\sigma} = (F(h_i))_{i\in I_\sigma}$, where $F := V_\psi f$, and obtain

$$||\sum_{i\in I_\sigma}\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}} S_\varphi^{-1}\left(\langle\varphi_i\circ\sigma, R(h,\cdot)\rangle\right)||_{L_{p,w}} \leq C\,||\left(\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}}\right)_{i\in I_\sigma}||_{\ell_{p,w}}$$

and together with Corollary 4.2 and the correspondence principle

$$||f||_{M_{p,w}} \leq C\,||\left(\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}}\right)_{i\in I_\sigma}||_{\ell_{p,w}}.$$

■

4.4 The Dual Spaces

The general coorbit space theory outlined above can also be used to derive atomic decompositions of the dual spaces $M'_{p,w}$ associated with $M_{p,w}$. Let us start with the following observation.

Theorem 4.4 *Let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{P},\sigma)$ and ψ a strictly admissible function. Then*

$$M'_{p,w} = M_{q,1/w} \qquad \textit{for} \quad 1<p<\infty, \quad 1/p+1/q=1.$$

Proof: The proof can be performed by following the lines of the proof of Theorem 4.9 in [11] and using the fact that for $1<p<\infty$ the space $L_{p,w}(X)$ is a reflexive Banach space. ■

By Theorem 4.4, atomic decompositions and Banach frames for $M'_{p,w}$ can be obtained by generalizing Theorem 4.1 and Theorem 4.2 to weighted spaces associated with $1/w$. Fortunately, the whole theory carries over without any essential difficulty.

Theorem 4.5 *Let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{P},\sigma)$ in $\mathcal{H}$ with strictly admissible function ψ. Let a compact neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$ be chosen such that* (4.1) *is satisfied with w replaced by $1/w$. Let $\mathcal{X}=(x_i)_{i\in\mathcal{I}}$ be a $\mathcal{U}$–dense and relatively separated family. Furthermore, suppose that condition* (3.6) *is satisfied and that for some compact neighborhood $\mathcal{Q}\subseteq\mathcal{U}$ of the identity in $\mathcal{G}$* (4.2) *holds for all $i\in\mathcal{I}_\sigma$. Finally, let us assume that our analyzing function ψ fulfills the following inequality*

$$\int_X \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{\mathcal{H}}| \frac{w(q^{-1}\sigma(l))}{w(\sigma(h))}\,d\mu(l) \leq \tilde{C}_{\mathcal{Q}}$$

with a constant $\tilde{C}_{\mathcal{Q}} < \infty$ *independent of* $h \in X$. *Then* $M_{p,\frac{1}{w}}$, $1 \le p \le \infty$, *has the following atomic decomposition: if* $f \in M_{p,\frac{1}{w}}$, $1 \le p \le \infty$, *then* f *can be represented as*

$$f = \sum_{i \in \mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi,$$

where the sequence of coefficients $(c_i)_{i \in I_\sigma} = (c_i(f))_{i \in I_\sigma} \in \ell_{p,\frac{1}{w}}$ *depends linearly on* f *and satisfies*

$$||(c_i)_{i \in I_\sigma}||_{\ell_{p,\frac{1}{w}}} \le A||f||_{M_{p,\frac{1}{w}}}.$$

If $(c_i)_{i \in I_\sigma} \in \ell_{p,\frac{1}{w}}$, *then* $f = \sum_{i \in \mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi$ *is contained in* $M_{p,\frac{1}{w}}$ *and*

$$||f||_{M_{p,\frac{1}{w}}} \le B||(c_i)_{i \in I_\sigma}||_{\ell_{p,\frac{1}{w}}}.$$

Theorem 4.6 *Impose the same assumptions as in Theorem 4.2 but with*

$$\int_X \sup_{q \in \mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle_{\mathcal{H}}| \frac{w(\sigma(h))}{w(q^{-1}\sigma(l))}\, d\mu(l) \le \tilde{C}_{\mathcal{Q}}$$

where $\tilde{C}_{\mathcal{Q}} < \infty$ *is a constant independent of* $h \in X$, *instead of* (4.7) *and with* w *replaced by* $1/w$ *in* (4.6). *Moreover, let us assume that* (3.6) *is satisfied. Then the set*

$$\{\psi_i := U(\sigma(h_i)^{-1})\psi : i \in \mathcal{I}_\sigma\}$$

is a Banach frame for $M_{p,\frac{1}{w}}$. *This means that*

i) $f \in M_{p,\frac{1}{w}}$ *if and only if* $(\langle f, \psi_i\rangle_{H'_{1,w} \times H_{1,w}})_{i \in \mathcal{I}_\sigma} \in \ell_{p,\frac{1}{w}}$;

ii) there exist two constants $0 < A' \le B' < \infty$ *such that*

$$A' \, \|f\|_{M_{p,\frac{1}{w}}} \le \|(\langle f, \psi_i\rangle_{H'_{1,w} \times H_{1,w}})_{i \in \mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} \le B' \, \|f\|_{M_{p,\frac{1}{w}}};$$

iii) there exists a bounded, linear reconstruction operator $\mathcal{S}$ *from* $\ell_{p,\frac{1}{w}}$ *to* $M_{p,\frac{1}{w}}$ *such that* $\mathcal{S}\left((\langle f, \psi_i\rangle_{H'_{1,w} \times H_{1,w}})_{i \in \mathcal{I}_\sigma}\right) = f$.

5 Application to the Sphere

In this section, we want to fill our technical consideration developed in the previous sections with live by giving an example. Following the lines of [3], we derive a generalized windowed Fourier transform on the spheres S^{n-1} and check that the proposed construction of modulation spaces and Banach frames works well for this setting. We start by establishing a suitable group representation for the Hilbert space $\mathcal{H} = L_2(S^{n-1})$. Having the usual windowed Fourier transform generated by translations and modulations in mind, Torresani suggested in [18] to choose the Euclidean group

$$\mathcal{G} := E(n) = SO(n) \ltimes \mathbb{R}^n,$$

i.e., the semidirect product of the special orthogonal group $SO(n)$ and $\mathbb{R}^n$ with group operation

$$(R,p)\circ(\tilde{R},\tilde{p}) = (R\tilde{R}, R\tilde{p}+p), \quad (R,p)^{-1} = (R^{-1}, -R^{-1}p).$$

As a natural analogue to the Schrödinger representation of the Weyl-Heisenberg group on $L_2(\mathbb{R}^n)$, we can define the continuous unitary representation U of $\mathcal{G}$ on $\mathcal{H}$

$$U(R,p)f(s) := e^{i<s,p>} f(R^{-1}s)\ ,$$

where $s \in S^{n-1}$. Since this representation is not square integrable, we are looking for suitable representations modulo a subgroup $\mathcal{P}$ of $\mathcal{G}$.

In order to keep the notation simple, we restrict ourselves to the case $\mathcal{H} = L_2(S^1) \cong L_2([-\pi,\pi])$. In this setting, $R \in SO(2)$ and $s \in S^1$ are given explicitly by

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad s = \begin{pmatrix} \sin\gamma \\ \cos\gamma \end{pmatrix}$$

and the representation U acts as

$$U(\theta,p_1,p_2)\psi(\gamma) = e^{i(p_1\sin\gamma + p_2\cos\gamma)}\psi(\gamma-\theta)\ .$$

To overcome the integrability problem, we use the subgroup $\mathcal{P} \cong \{(0,0,p_2) \in \mathcal{G}\}$ together with the flat section $\sigma(\theta,p_1) = (\theta,p_1,0)$. Then the following lemma proved in [18] ensures strictly square integrability of U mod $(\mathcal{P},\sigma)$.

Lemma 5.1 *Assume that the function $\psi \in L_2([-\pi,\pi])$ is such that* supp $\psi \subset [-\pi/2,\pi/2]$ *and*

$$2\pi \int_{-\pi/2}^{\pi/2} \frac{|\psi(\gamma)|^2}{\cos\gamma} d\gamma = 1\ .$$

Then the map V_ψ defined by (2.2) *is an isometry.*

In the following, we choose the admissible function

$$\psi(x) = \cos^6 x \cdot \chi_{[-\pi/2,\pi/2]}(x).$$

In order to construct properly defined modulation spaces we have to establish the fundamental property (3.1) of our kernel R. Moreover, for the construction of Banach frames in our weighted coorbit spaces $M_{p,w}$, we have to verify the related property (4.8). It has been shown in [3] that the kernel R can be rewritten as

$$R(l,h) = \hat{F}_{\theta,p_l}(-p_h)\ ,$$

where $h = (\theta_h, p_h, 0), l = (\theta_l, p_l, 0) \in X$, $\theta = \theta_h - \theta_l$ and

$$F_{\theta,p_l}(t) := e^{-ip_l\sin(\arcsin t-\theta)}\psi(\arcsin t-\theta)\overline{\psi}(\arcsin t)/\sqrt{1-t^2}.$$

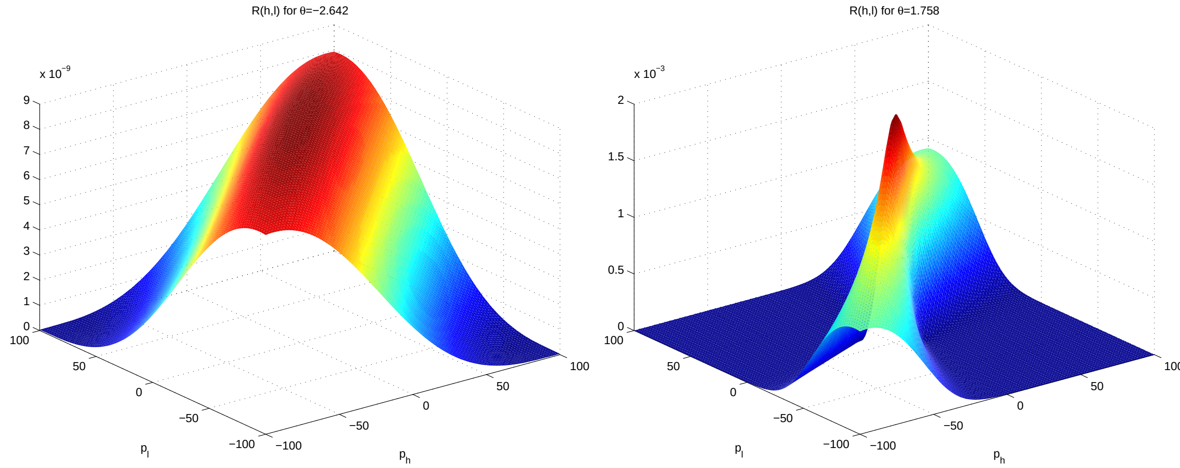

Figure 1: Left: $|\hat{F}_{\theta,p_l}(-p_h)|$ for $\theta = -2.642$, right: $|\hat{F}_{\theta,p_l}(-p_h)|$ for $\theta = 1.758$

The plots of $|R(h,l)| = |\hat{F}_{\theta,p_l}(-p_h)|$ for two values of θ in Figure 1 describe the typical decay behaviour of R.

In analogy to the classical modulation spaces on the Euclidean plane, we consider specific weight functions of the form $w(R,p) = (1+\|p\|)^s$, i.e., the modulation spaces are generalized Bessel–potential spaces. Since the invariant measure $d\mu(h)$ of X is given by $dp_h d\theta_h$, we obtain that

$$\int_X |R(h,l)| \frac{w(\sigma(h))}{w(\sigma(l))}\, d\mu(h) = \int_{-\pi}^{\pi} \int_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)| \frac{(1+|p_h|)^s}{(1+|p_l|)^s}\, dp_h\, d\theta_h.$$

Regarding that the outer integration is over a finite interval, we see that property (3.1) is equivalent to

$$\int_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)| \frac{(1+|p_h|)^s}{(1+|p_l|)^s}\, dp_h < C, \tag{5.1}$$

with some constant C independent of p_l and θ. Similarly, we conclude that property (4.8) is equivalent to

$$\int_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)| \frac{(1+|p_h|)^s}{(1+|p_l|)^s}\, dp_l \leq C \tag{5.2}$$

with some constant C independent of p_h and θ. These properties are confirmed numerically and the results are presented in the Figures 2 – 4 for $s = 0.5$. Figure 2 shows the approximated values of $\int_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)|(1+|p_h|)^{0.5}/(1+|p_l|)^{0.5}\, dp_h$ as functions of p_l and Figure 3 the approximated values of $\int_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)|(1+|p_h|)^{0.5}/(1+|p_l|)^{0.5}\, dp_l$ as functions of p_h. Finally, in Figure 4, we have displayed $\max_{p_l} \int |\hat{F}_{\theta,p_l}(p_h)(1+|p_h|)^{0.5}/(1+|p_l|)^{0.5}\, dp_h$ and $\max_{p_h} \int |\hat{F}_{\theta,p_l}(p_h)(1+|p_h|)^{0.5}/(1+|p_l|)^{0.5}\, dp_l$ for all $\theta \in [-\pi, \pi]$. These results clearly show that conditions (5.1) and (5.2) are satisfied.

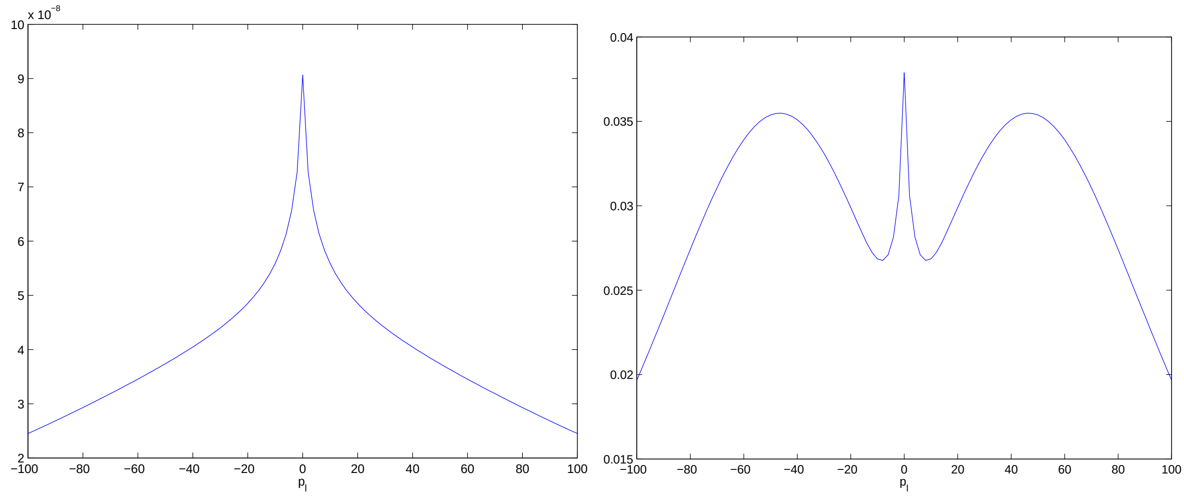

Figure 2: $\int_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)|(1+|p_h|)^{0.5}/(1+|p_l|)^{0.5}\,dp_h$, left: $\theta = -2.642$, right: $\theta = 1.758$

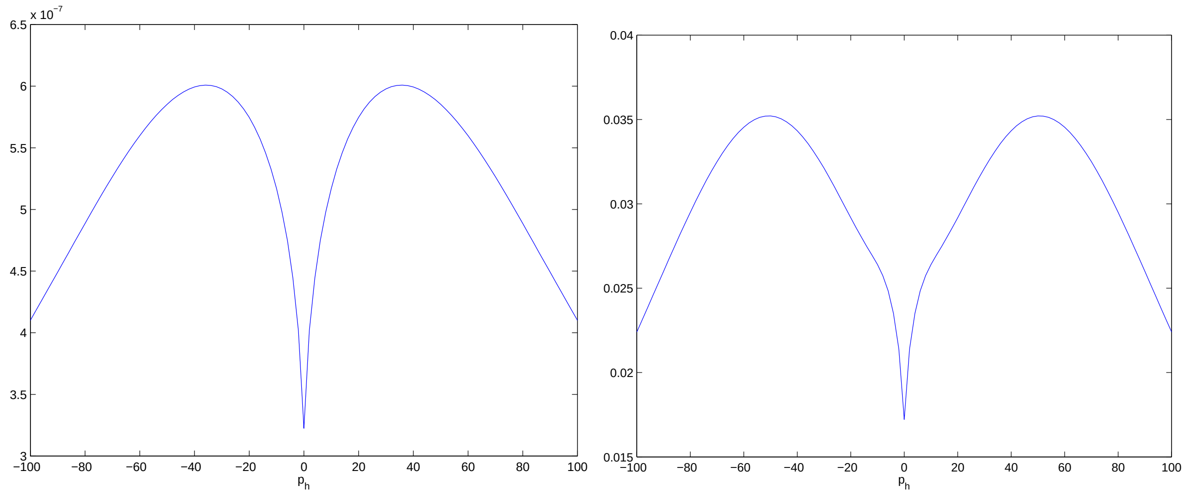

Figure 3: $\int_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)|(1+|p_h|)^{0.5}/(1+|p_l|)^{0.5}\,dp_l$, left: $\theta = -2.642$, right: $\theta = 1.758$

For the construction of Banach frames in $M_{p,w}$ we choose the neighborhood $\mathcal{U} := [-\pi/N, \pi/N] \times [-\pi/M, \pi/M] \times [-\pi/M, \pi/M]$ of the identity and a $\mathcal{U}$–dense set $\mathcal{X} := (x_{n,m})_{(n,m)\in\mathcal{I}}$ with $x_{n,m} = (\theta_n, p_m, q_m)$. Then the assumptions concerning $\mathrm{osc}_{\mathcal{U}}$ in Theorem 4.1 and Theorem 4.2, respectively, can be verified directly by slightly modifying the steps in [3] with respect to the additional weight function. Finally, note that property (4.2) has already been proved in [3] for a suitable set $\mathcal{Q}$.

A Appendix

In this section, we want to collect some basic facts that were needed before.

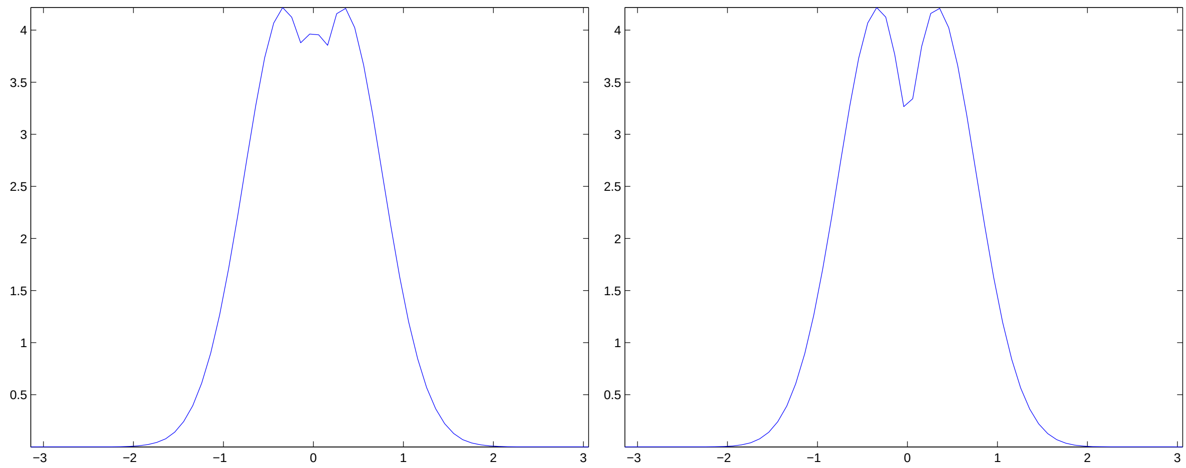

Figure 4: Maximum plot for all $\theta \in [-\pi, \pi]$; left: $\max_{p_l} \int_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)|(1+|p_h|)^{0.5}/(1+|p_l|)^{0.5}\, dp_h$, right: $\max_{p_h} \int_{\mathbb{R}} |\hat{F}_{\theta,p_l}(p_h)|(1+|p_h|)^{0.5}/(1+|p_l|)^{0.5}\, dp_l$.

Lemma A.1 *Let D be a reflexive Banach space, E a normed space and $T : D \to E$ a linear, injective, bounded mapping with dense image in E. Then $T' : E' \to D'$ defined by $T'(f) = f \circ T$, $f \in E'$ is a linear, injective, bounded mapping with dense image in D'.*

Proof: The operator T' is injective because $T'(e_1') = T'(e_2')$ for some $e_1', e_2' \in E'$ means by definition of T' that $e_1'(Td) = e_2'(Td)$ for all $d \in D$. By the density of $T(D)$ in E this implies that $e_1' = e_2'$.

The operator T' has a dense image in D' by the following argument: assume that $T'(E')$ is not dense in D'. Then, by the Hahn–Banach Theorem, there exists $u \in D''$, $u \neq 0$ such that $u(F) = 0 \quad \text{for all} \quad F \in T'(E')$. Since D is reflexive, there exists $d \in D$ such that

$$u(F) = F(d) \quad \text{for all} \quad F \in D'. \tag{A.1}$$

Consequently, we obtain for all $e' \in E'$ that $e'(T(d)) = u(e' \circ T) = u(T'(e')) = 0$. Thus, $T(d) = 0$ which implies by injectivity of T that $d = 0$. But, by (A.1), this implies the contradiction $u = 0$.

Finally, the continuity of T' follows by

$$\|T'(e')\|_{D'} = \sup_{d \in D} \frac{|T'(e')(d)|}{\|d\|_D} = \sup_{d \in D} \frac{|(e' \circ T)(d)|}{\|d\|_D}$$

and since $\|T(d)\|_E \leq \|T\|_{D \to E} \|d\|_D$ for all $d \in D$ and $T(D)$ is dense in E

$$\|T'(e')\|_{D'} \leq \|T\|_{D \to E} \sup_{x \in T(D)} \frac{|e'(x)|}{\|x\|_E} = \|T\|_{D \to E} \|e'\|_{E'}.$$

■

Next, we extend the classical Young inequality, see, e.g., [14], p. 185, Theorem 6.18, to weighted L_p–spaces.

Theorem A.1 (**Weighted Young Inequality**) *Let $(Z, \mathcal{A}, \eta)$ and $(Y, \mathcal{B}, \zeta)$ be σ–finite measure spaces, let K be an $\mathcal{A}\otimes\mathcal{B}$–measurable function on $Z \times Y$, and let w be a positive weight function. Suppose that K satisfies the following conditions*

$$\int_Z |K(x,y)| \frac{w(x)}{w(y)} \, d\eta(x) \leq C_K$$

for a.e. $y \in Y$ and

$$\int_Y |K(x,y)| \frac{w(x)}{w(y)} \, d\zeta(y) \leq C_K$$

for a.e. $x \in Z$. If $f \in L_{p,w}$, $1 \leq p \leq \infty$, then the integral

$$Tf(x) = \int_Y K(x,y) f(y) \, d\zeta(y)$$

converges absolutely for a.e. $x \in Z$, the function Tf thus defined is in $L_{p,w}$ and

$$\|Tf\|_{L_{p,w}} \leq C_K \|f\|_{L_{p,w}}.$$

For reader's convenience, we include a complete proof here. However, let us remark that the result can also be derived by applying the classical Young inequality as outlined in Folland [14] to suitable weighted functions and kernels.

Proof: To show that the operator T is bounded we apply the assumptions of Theorem A.1 and the Hölder inequality with $1/p + 1/q = 1$ as follows:

$$\begin{aligned}
\|Tf\|^p_{L_{p,w}} &= \int | \int K(x,y) f(y) d\zeta(y)|^p \, w^p(x) \, d\eta(x) \\
&\leq \int \left(\int |K(x,y)| \frac{1}{w(y)^{1/p+1/q}} \, |f(y)| \, w(y) \, d\zeta(y) \right)^p \, w^p(x) \, d\eta(x) \\
&\leq \int \left(\int |K(x,y)| \frac{1}{w(y)} \, |f(y)|^p \, w^p(y) \, d\zeta(y) \right) \left(\int |K(x,y)| \frac{1}{w(y)} \, d\zeta(y) \right)^{p/q} \\
&\quad \times w^p(x) \, d\eta(x) \\
&\leq C_K^{p/q} \int \int |K(x,y)| \frac{1}{w(y)} \, |f(y)|^p \, w^p(y) \, d\zeta(y) \, w(x)^{p-p/q} \, d\eta(x) \\
&= C_K^{p/q} \int |f(y)|^p \, w^p(y) \int |K(x,y)| \frac{w(x)}{w(y)} \, d\eta(x) \, d\zeta(y) \\
&\leq C_K^p \|f\|^p_{L_{p,w}} \, . \qquad \blacksquare
\end{aligned}$$

B Appendix

In order to establish the frame bounds, we need a variant of the Riesz–Thorin interpolation theorem for the case of weighted L_p–spaces. For $p_0, p_1 < \infty$, the desired result is essentially a special case of the Stein–Weiss interpolation theorem, see, e.g., [2], Corollary 5.5.4, for details. However, for our approach we definitely need the corresponding result for $p_0 = 1, p_1 = \infty$. The resulting theorem is stated and proved below. It might be already known to the specialists, however, in this special form, it was not found in the literature.

The proof is based on complex interpolation. Therefore we start by briefly recalling the basic setting. For further information concerning real and complex interpolation, the reader is, e.g., refered to [2] and [17]. Let A_0 and A_1 be two complex Banach spaces. Then (A_0, A_1) is called an *interpolation couple* if there exists a linear complex Hausdorff space such that both A_0 and A_1 are linearly and continuously embedded in this space. Then $A_0 \cap A_1$ with norm $\|a\|_{A_0\cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A1}\}$ and $A := A_0 + A_1$ with norm $\|a\|_{A_0+A_1} = \inf_{a=a_0+a_1} \{\|a\|_{A_0}, \|a\|_{A1}\}$ are also complex Banach spaces. Let

$$\mathcal{S} := \{z \in \mathbb{C} : \quad 0 < \Re z < 1\}$$

be a strip in the complex plane. The collection $\mathcal{F}$ of all functions $f(z)$ defined on $\overline{\mathcal{S}}$ with values in A with the two properties

i) $f(z)$ is continuous in $\overline{\mathcal{S}}$ and analytic in $\mathcal{S}$ with

$$\sup_{z\in\overline{\mathcal{S}}} \|f(z)\|_A < \infty,$$

ii) $f(it) \in A_0$ and $f(1+it) \in A_1$, with $t \in \mathbb{R}$, are continuous in the respective Banach spaces and

$$\|f\|_{\mathcal{F}} := \max\{\sup_t \|f(it)\|_{A_0}, \sup_t \|f(1+it)\|_{A_1}\} < \infty$$

is again a Banach space.

For a given interpolation couple (A_0, A_1) and $\theta \in (0,1)$, the space $(A_0, A_1)_{[\theta]}$ is defined as

$$(A_0, A_1)_{[\theta]} := \{a \in A : \text{ there exists } f(z) \in \mathcal{F} \text{ with } f(\theta) = a\}.$$

Equipped with the norm

$$\|a\|_{[\theta]} := \inf\{\|f\|_{\mathcal{F}} : \ f(\theta) = a\},$$

$(A_0, A_1)_{[\theta]}$ becomes a Banach space which has the following *interpolation property*:

Theorem B.1 *Let (A_0, A_1) and (B_0, B_1) be two interpolation couples and let T be a linear operator from $A_0 + A_1$ into $B_0 + B_1$ such that its restriction to A_j is a bounded linear operator from A_j into B_j, with norm $\leq M_j, j = 0, 1$. Then for any $\theta \in (0,1)$, the restriction of T to (A_0, A_1) is a bounded linear operator from $(A_0, A_1)_{[\theta]}$ into $(B_0, B_1)_{[\theta]}$ with norm $\leq M_0^{1-\theta} M_1^{\theta}$.*

Theorem B.1 is the main ingredient for the proof of Theorem B.2. For technical reasons, we shall also need the so–called three line theorem, see [2], page 4 for details.

Lemma B.1 *(The three line theorem) Assume that $F(z)$ is analytic on $\mathcal{S}$ and bounded and continuous on $\overline{\mathcal{S}}$. If*

$$|F(it)| \leq N_0, \qquad |F(1+it)| \leq N_1, \qquad -\infty < t < \infty,$$

then we have for $\theta \in [0,1]$ that

$$|F(\theta + it)| \leq N_0^{1-\theta} N_1^{\theta}, \qquad -\infty < t < \infty.$$

Now we are ready to establish the desired interpolation result with respect to $L_{1,w}$ and $L_{\infty,w}$.

Theorem B.2 *Let T be a bounded linear operator from $L_{1,w}$ into $\ell_{1,w}$ with norm M_1 and from $L_{\infty,w}$ into $\ell_{\infty,w}$ with norm M_∞. Then, for any $1 < p < \infty$, the operator T is also a bounded from $L_{p,w}$ into $\ell_{p,w}$ with norm $M_1^{1/p} M_\infty^{(p-1)/p}$.*

Proof: According to Theorem B.1, it remains to show that

$$(L_{1,w}, L_{\infty,w})_{[\theta]} = L_{p,w} \quad \text{and} \quad (\ell_{1,w}, \ell_{\infty,w})_{[\theta]} = \ell_{p,w} \ , \tag{B.1}$$

where $1/p = 1 - \theta$. We only prove the first statement in (B.1), the second one follows analogously. We have to show that

$$\|a\|_{[\theta]} = \|a\|_{(L_{1,w}, L_{\infty,w})_{[\theta]}} = \|a\|_{L_{p,w}} \ .$$

We start with the proof of $\|a\|_{[\theta]} \leq \|a\|_{L_{p,w}}$. Without loss of generality we may assume that $\|a\|_{L_{p,w}} = 1$. For our purposes, it is convenient to define f as follows

$$f(z) := w(x)^{p(1-z)-1} \exp(\varepsilon(z^2 - \theta^2)) |a(x)|^{p(1-z)} \frac{a(x)}{|a(x)|} \ .$$

We observe that f is an analytic function on the strip $\mathcal{S}$ with $f(\theta) = a$. In order to compute $\|a\|_{[\theta]}$ we note that

$$\|f\|_{\mathcal{F}} = \max\{\sup_t \|f(it)\|_{L_{1,w}}, \sup_t \|f(1+it)\|_{L_{\infty,w}}\} \ . \tag{B.2}$$

For $\|f(it)\|_{L_{1,w}}$, we obtain

$$\begin{aligned}
\|f(it)\|_{L_{1,w}} &= \int w(x) |w(x)^{p(1-it)-1} \exp(\varepsilon(-t^2 - \theta^2)) |a(x)|^{p(1-it)} \frac{a(x)}{|a(x)|}| dx \\
&= \exp(\varepsilon(-t^2 - \theta^2)) \int |a(x)|^p w(x)^p \, dx \\
&= \exp(\varepsilon(-t^2 - \theta^2) \|a\|_{L_{p,w}}^p = \exp(\varepsilon(-t^2 - \theta^2).
\end{aligned}$$

Consequently, for some suitable ε,

$$\sup_t \|f(it)\|_{L_{1,w}} = \exp(-\varepsilon\theta^2) \leq 1 \ . \tag{B.3}$$

The $L_{\infty,w}$–norm of $f(1+it)$ can be estimated as

$$\begin{aligned}\|f(1+it)\|_{L_{\infty,w}} &= \sup_x w(x)|w(x)^{p(1-(1+it))-1}\exp(\varepsilon((1+it)^2-\theta^2))|a(x)|^{p(1-(1+it))}\frac{a(x)}{|a(x)|}| \\ &= \exp(\varepsilon(1-t^2-\theta^2)) \le \exp(\varepsilon)\ . \end{aligned} \tag{B.4}$$

Combining (B.3) and (B.4) we obtain by (B.2)

$$\|f\|_{\mathcal{F}} \le \exp(\varepsilon) \to 1 \quad \text{for} \quad \varepsilon \to 0\ ,$$

and taking the infimum yields

$$\|a\|_{[\theta]} \le \|a\|_{L_{p,w}}\ \text{, i.e.,}\ \ L_{p,w} \subset (L_{1,w}, L_{\infty,w})_{[\theta]}\ .$$

The next step is to show $\|a\|_{L_{p,w}} \le \|a\|_{[\theta]}$. Without loss of generality we may again assume that $\|a\|_{[\theta]} = 1$. Then we have

$$\|a\|_{L_{p,w}} = \sup\{|\langle a,b\rangle_w| :\ \|b\|_{L'_{p,w}} = 1\},$$

where, for $1 \le p < \infty$, the dual pairing can be written as

$$\langle a,b\rangle_w := \int a(x)b(x)w(x)^p\,dx\ .$$

We define

$$F(z) := \langle f(z), g(z)\rangle_w$$

for some $f \in \mathcal{F}$ satisfying $f(\theta) = a$ and g given by

$$g(z) := w(x)^{1-p(1-z)}\exp(\varepsilon(z^2-\theta^2))|b(x)|^{pz/(p-1)}\frac{b(x)}{|b(x)|}$$

for some $b \in L'_{p,w}$ with $\|b\|_{L'_{p,w}} = 1$. We want to estimate $F(z)$ by means of Lemma B.1. Since $\|a\|_{[\theta]} = 1$ we can find $f \in \mathcal{F}$ with $f(\theta)) = a$ such that $\|f(it)\|_{L_{1,w}} \le 1+\varepsilon$ and $\|f(1+it)\|_{L_{\infty,w}} \le 1+\varepsilon$ for all $\varepsilon > 0$. Any such function f provides us with suitable bounds for $|F(it)|$ and $|F(1+it)|$. Indeed,

$$\begin{aligned}|F(it)| &= |\int f(it)g(it)w(x)^p\,dx| \\ &\le \int |f(it)||w(x)^{1-p(1-it)}|w(x)^p\,dx\,\exp(\varepsilon(-t^2-\theta^2)) \\ &\le \int |f(it)|w(x)\,dx\,\exp(\varepsilon(-t^2-\theta^2)) \\ &\le \|f(it)\|_{L_{1,w}}\exp(\varepsilon(-t^2-\theta^2)) \\ &\le (1+\varepsilon)\exp(-\varepsilon\theta^2) \le \exp(\varepsilon) =: N_0\end{aligned}$$

and

$$\begin{aligned}
|F(1+it)| &= |\int g(1+it)f(1+it)w(x)^p\,dx| \\
&\leq \|f(1+it)\|_{L_{\infty,w}} \int |b(x)|^{p(1+it)/(p-1)} w(x)^{p(1+it)}\,dx \exp(\varepsilon(1-t^2-\theta^2)) \\
&\leq (1+\varepsilon) \int |b(x)|^{p/(p-1)} w(x)^p\,dx \exp(\varepsilon)\exp(\varepsilon(-t^2-\theta^2)) \\
&\leq \exp(2\varepsilon) =: N_1 \ .
\end{aligned}$$

Hence, by using Lemma B.1,

$$|F(\theta+it)| \leq \exp(2\varepsilon) \quad \text{for all } 0 \leq \theta \leq 1 \ .$$

Consequently,

$$|\langle a, b\rangle_w| \leq |F(\theta)| \leq \exp(2\varepsilon) \ ,$$

that is, $\|a\|_{L_{p,w}} \leq 1$ and therefore $(L_{1,w}, L_{\infty,w})_{[\theta]} \subset L_{p,w}$. ■

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