# Preconditioners for nondefinite Hermitian Toeplitz systems

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**Abstract**

This paper is concerned with the construction of circulant preconditioners for Toeplitz systems arising from a piecewise continuous generating function with sign changes.
If the generating function is given, we prove that for any $\varepsilon > 0$, only $\mathcal{O}(\log N)$ eigenvalues of our preconditioned Toeplitz systems of size $N \times N$ are not contained in $[-1-\varepsilon, -1+\varepsilon] \cup [1-\varepsilon, 1+\varepsilon]$. The result can be modified for trigonometric preconditioners. We also suggest circulant preconditioners for the case that the generating function is not explicitly known and show that only $\mathcal{O}(\log N)$ absolute values of the eigenvalues of the preconditioned Toeplitz systems are not contained in a positive interval on the real axis.
Using the above results, we conclude that the preconditioned minimal residual method requires only $\mathcal{O}(N \log^2 N)$ arithmetical operations to achive a solution of prescribed precision if the spectral condition numbers of the Toeplitz systems increase at most polynomial in $N$. We present various numerical tests.

## 1 Introduction

Let $\mathcal{L}_{2\pi}$ be the space of $2\pi$-periodic Lebesgue integrable real-valued functions and let $C_{2\pi}$ be the subspace of $2\pi$-periodic real-valued continuous functions with norm

$$\|f\|_\infty := \max_{t \in [-\pi,\pi]} |f(t)| \qquad (f \in C_{2\pi}).$$

The Fourier coefficients of $f \in \mathcal{L}_{2\pi}$ are given by

$$a_k = a_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \mathrm{e}^{-\mathrm{i}kt} \,\mathrm{d}t \qquad (k \in \mathbb{Z})$$

and the sequence $\{\boldsymbol{A}_N(f)\}_{N=1}^{\infty}$ of $(N,N)$-Toeplitz matrices generated by $f$ is defined by

$$\boldsymbol{A}_N = \boldsymbol{A}_N(f) := (a_{j-k}(f))_{j,k=0}^{N-1}.$$

Since $f \in \mathcal{L}_{2\pi}$ is real-valued the matrices $\boldsymbol{A}_N(f)$ are Hermitian.
In this paper, we are interested in the iterative solution of Toeplitz systems

$$\boldsymbol{A}_N(f)\, \boldsymbol{x} = \boldsymbol{b}, \qquad (1.1)$$ 1.1

where we allow the generating function $f \in \mathcal{L}_{2\pi}$ to vary in sign. To be more precise, we are looking for good preconditioning strategies so that Krylov space methods applied to the preconditioned system converge in a few number of iteration steps. Note that by the Toeplitz structure of $\boldsymbol{A}_N$ each iteration step requires only $\mathcal{O}(N \log N)$ arithmetical operations by using fast Fourier transforms.

Up to now iterative methods for Toeplitz systems with generating functions having different signs were only considered in [14, 16] and in connection with non-Hermitian systems in [3, 2]. In [3], we have constructed circulant preconditioners for non-Hermitian Toeplitz matrices with known generating function of the form

$$f = p\,h\,,$$

where $p$ is an arbitrary trigonometric polynomial and $h$ is a function from the Wiener class with $|h| > 0$. We proved that the preconditioned matrices have *singular values* properly clustered at 1. Then, if the *spectral condition number* of $\boldsymbol{A}_N(f)$ fulfills $\kappa_2(\boldsymbol{A}_N(f)) = N^\alpha$, the *conjugate gradient method* (CG) applied to the normal equation requires only $\mathcal{O}(\log N)$ iteration steps to produce a solution of fixed precision. However, in general nothing can be said about the eigenvalues of the preconditioned matrix.
In this paper, we consider real-valued functions $f \in \mathcal{L}_{2\pi}$ of the form

$$f = p_s h, \tag{1.2}$$

where

$$p_s(t) := \prod_{j=1}^{\mu} (2 - 2\cos(t - t_j))^{s_j}, \;\; s := \sum_{j=1}^{\mu} s_j \tag{1.3}$$

is a trigonometric polynomial with a finite number of zeros $t_j \in [-\pi, \pi)$ $(j = 1, \ldots, \mu)$ of even order $2s_j$ and where $h \in \mathcal{L}_{2\pi}$ is a piecewise continuous function with simple discontinuities at $\xi_j$ $(j = 1, \ldots, \nu)$, i.e. there exist $h(\xi_j \pm 0)$ and $h(\xi_j + 0) - h(\xi_j - 0) = \alpha_j \neq 0$. For simplicity let $h(\xi_j) = (h(\xi_j - 0) + h(\xi_j + 0))/2$. Further, we assume that

$$\overline{\{|h(t)| : t \in [-\pi, \pi);\, |h(t)| > 0\}} \subseteq [h_-, h_+]\,, \tag{1.4}$$

where $0 < h_- \leq h_+ < \infty$.
A similar setting was also considered in [14]. S. Serra Capizzano suggested the application of of band-Toeplitz preconditioners $\boldsymbol{A}_N(p_s)$ in combination with CG applied to the normal equation. He proved, beyond a more general result which can not directly be used for preconditioning, that at most $o(N)$ eigenvalues of the preconditioned matrix $\boldsymbol{A}_N(p_s)^{-1}\boldsymbol{A}_N(f)$ have absolute values not contained in a positive interval on the real axis.
A result with $o(N)$ outlyers was also obtained in [15], where the application of preconditioned GMRES was examined.
In the following, we construct circulant preconditioners for the *minimal residual method* (MINRES). Note that preconditioned MINRES avoids the translation of the original system to the normal equation but requires Hermitian positive definite preconditioners. Then, the preconditioned matrices are again Hermitian, so that the absolute values of their eigenvalues coincide with their singular values. If the generating function is given, we prove that for any $\varepsilon > 0$, only $\mathcal{O}(\log N)$ singular values of the preconditioned matrices are not contained in $[1-\varepsilon, 1+\varepsilon]$. We also construct circulant preconditioners for the case that the generating function of the Toeplitz matrices is not explicitly known. For this, we use positive reproducing kernels with special properties previously applied by the authors in [12, 5] and show that $\mathcal{O}(\log N)$ singular values of the preconditioned matrices are not contained in a positive interval on the real axis. Then, if in addition $\kappa_2(\boldsymbol{A}_N(f)) = N^\alpha$, preconditioned MINRES converges in at most $\mathcal{O}(\log N)$ iteration steps. In summary, the proposed algorithm requires only $\mathcal{O}(N \log^2 N)$ arithmetical operations.

This paper is organized as follows: In Section 2, we introduce circulant preconditioners for (1.1) under the assumption that the generating function of the sequence of Toeplitz matrices is known and prove clustering results for the eigenvalues of the preconditioned matrices. Section 3 deals with the construction of preconditioners if the generating function of the Toeplitz matrices is not explicitly known. In Section 4, we modify the results of Section 2 with respect to trigonometric preconditioners. The convergence of MINRES applied to our preconditioned Toeplitz systems is considered in Section 5. Finally, we present numerical results in Section 6.

## 2 Circulant preconditioners involving generating functions

First we introduce some basic notation. By $\boldsymbol{R}_N(M)$ we denote arbitrary $(N,N)$-matrices of rank *at most* $M$. Let $\boldsymbol{M}_N(g)$ be the circulant $(N,N)$-matrix

$$\boldsymbol{M}_N(g) := \boldsymbol{F}_N \operatorname{diag}\left(g\left(\frac{2\pi l}{N}\right)\right)_{l=0}^{N-1} \boldsymbol{F}_N^*,$$

where $\boldsymbol{F}_N$ denotes the $N$-th *Fourier matrix*

$$\boldsymbol{F}_N := \frac{1}{\sqrt{N}} \left(\mathrm{e}^{-2\pi \mathrm{i} jk/N}\right)_{j,k=0}^{N-1}$$

and where $\boldsymbol{F}^*$ is the transposed complex conjugate matrix of $\boldsymbol{F}$. For a trigonometric polynomial $q(t) := \sum_{k=-n_1}^{n_2} q_k \mathrm{e}^{\mathrm{i}kt}$, the matrices $\boldsymbol{A}_N(q)$ and $\boldsymbol{M}_N(q)$ are related by

$$\boldsymbol{A}_N(q) = \boldsymbol{M}_N(q) + \boldsymbol{R}_N(n_1+n_2) \tag{2.1}$$

(see [10]). For a function $g$ with a finite number of zeros we define the set $I_N(g)$ by

$$I_N(g) := \{l = 0, \ldots, N-1 : \; g\left(\frac{2\pi l}{N}\right) \neq 0\}.$$

and the points $x_{N,l}(g)$ $(l = 0, \ldots, N-1)$ by

$$x_{N,l}(g) := \begin{cases} \frac{2l\pi}{N} & \text{if} \quad l \in I_N(g)\,, \\ \frac{2\tilde{l}\pi}{N} & \text{otherwise}\,, \end{cases}$$

where $\tilde{l} \in \{0, \ldots, N-1\}$ is the next higher index to $l$ so that $\tilde{l} \in I_N(g)$. For $N$ large enough we can simply choose $\tilde{l} = l+1 \mod N$. By $\boldsymbol{M}_{N,g}(f)$ we denote the circulant matrix

$$\boldsymbol{M}_{N,g}(f) := \boldsymbol{F}_N \operatorname{diag}\left(f(x_{N,l}(g))\right)_{l=0}^{N-1} \boldsymbol{F}_N^*\,. \tag{2.2}$$

If $g$ has $m$ zeros, then we have by construction that

$$\boldsymbol{M}_N(f) \;=\; \boldsymbol{M}_{N,g}(f) \;+\; \boldsymbol{R}_N(m)\,. \tag{2.3}$$

Assume now that the sequence $\{\boldsymbol{A}_N(f)\}_{N=1}^{\infty}$ of nonsingular Toeplitz matrices is generated by a known piecewise continuous function $f \in \mathcal{L}_{2\pi}$ of the form (1.2) – (1.4). Then we suggest the Hermitian positive definite circulant matrix $\boldsymbol{M}_{N,f}(|f|)$ as preconditioner for MINRES.

We examine the distribution of the eigenvalues of $\boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}}\boldsymbol{A}_N(f)\boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}}$.

The following theorem is Lemma 10 of [18] written with respect to our notation.

T2.1 **Theorem 2.1** *Let $h \in \mathcal{L}_{2\pi}$ be a piecewise continuous function having only simple discontinuities at $\xi_j \in [-\pi, \pi)$ $(j = 1, \ldots, \nu)$. By $\mathcal{F}_N$ we denote the* Fejér kernel

$$\begin{aligned} \mathcal{F}_N(t) \quad &:= \quad \sum_{k=-(N-1)}^{N-1} \left(1 - \left|\frac{k}{N}\right|\right) \mathrm{e}^{ikt} = 1 + 2\sum_{k=1}^{N-1}\left(1 - \frac{k}{N}\right)\cos kt \qquad (2.4) \\ &= \quad \begin{cases} \frac{1}{N}\left(\sin\left(\frac{Nt}{2}\right) / \sin\left(\frac{t}{2}\right)\right)^2 & t \neq 0, \\ \frac{1}{N} & t = 0 \end{cases} \qquad (2.5) \end{aligned}$$

f1 f2

*and by $\mathcal{F}_N * h$ the* cyclic convolution *of $\mathcal{F}_N$ and $h$. Then, for any $\varepsilon > 0$, there exist constants $0 < c_1 \leq c_2 < \infty$ independent of $N$ so that the number $\nu(\varepsilon; \boldsymbol{A}_N)$ of eigenvalues of $\boldsymbol{A}_N(h) - \boldsymbol{M}_N(\mathcal{F}_N * h)$ with absolute value exceeding $\varepsilon$ can be estimated by*

$$c_1 \log\, (N) \leq \nu(\varepsilon; \boldsymbol{A}_N) \leq c_2 \log\, (N).$$

With other words, we have by Theorem 2.1 that

$$\boldsymbol{A}_N(h) = \boldsymbol{M}_N(\mathcal{F}_N * h) + \boldsymbol{V}_N + \boldsymbol{U}_N, \qquad (2.6)$$

3.1

where $\boldsymbol{V}_N$ is a matrix of spectral norm $\leq \varepsilon$ and where

$$c_1 \log N \leq \text{ rank } (\boldsymbol{U}_N) \leq c_2 \log N.$$

Using Theorem 2.1, we can prove the following lemma.

L2.2 **Lemma 2.2** *Let $f = p_s h \in \mathcal{L}_{2\pi}$ be given by (1.2) – (1.4). Then, for any $\varepsilon > 0$ and sufficiently large $N$, the number of singular values of $\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{A}_N(h)\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}$ which are not contained in the interval $[1-\varepsilon, 1+\varepsilon]$ is $\mathcal{O}(\log N)$.*

**Proof.** By (2.6) and since the eigenvalues of $\boldsymbol{M}_{N,f}(|h|)$ are restricted from below by $h_-$, it remains to show that for any $\varepsilon > 0$ and sufficiently large $N$, except for $\mathcal{O}(\log N)$ eigenvalues, all eigenvalues of $\boldsymbol{M}_{N,f}(|h|)^{-1}\boldsymbol{M}_N(\mathcal{F}_N * h)$ have absolute values in $[1-\varepsilon, 1+\varepsilon]$. Indeed we will prove that there are only $\mathcal{O}(1)$ outlyers.
For this we follow mainly the lines of proof of Gibb's phänomenon. Without loss of generality we assume that $h \in \mathcal{L}_{2\pi}$ has only one jump at $\xi_1 = 0$ of height $\alpha_1$.
First we examine $\mathcal{F}_N * g$, where $g$ is given by

$$g(x) := \begin{cases} \frac{1}{2}(\pi - x) & x \in (0, \pi), \\ \frac{1}{2}(x - \pi) & x \in (-\pi, 0), \\ 0 & x = 0. \end{cases}$$

By (2.4) and since $g$ has Fourier series

$$g(x) \sim \sum_{k=1}^{\infty} \frac{1}{k} \sin kx$$

we obtain

$$\int_0^x \mathcal{F}_N(t)\,\mathrm{d}t = x + 2\sum_{k=1}^{N-1} \left(\frac{1}{k} - \frac{1}{N}\right) \sin kx \;\; = x + 2\,(\mathcal{F}_N * g)(x)$$

and further by (2.5)

$$\begin{aligned}
(\mathcal{F}_N * g)(x) &= \frac{1}{2N}\int_0^x \left(\frac{\sin\frac{Nt}{2}}{\sin\frac{t}{2}}\right)^2 \mathrm{d}t - \frac{x}{2} \\
&= \frac{1}{2N}\int_0^x \left(\frac{\sin\frac{Nt}{2}}{\frac{t}{2}}\right)^2 \mathrm{d}t + \frac{1}{2N}\int_0^x \left(\frac{1}{(\sin\frac{t}{2})^2} - \frac{1}{(\frac{t}{2})^2}\right)\left(\sin\frac{Nt}{2}\right)^2 \mathrm{d}t - \frac{x}{2} \\
&= \int_0^{\frac{Nx}{2}} \left(\frac{\sin t}{t}\right)^2 \mathrm{d}t + \mathcal{O}(N^{-1}) - \frac{x}{2}
\end{aligned}$$

and by partial integration and definition of $g$

$$(\mathcal{F}_N * g)(x) - g(x) = \frac{-(\sin\frac{Nx}{2})^2}{\frac{Nx}{2}} + \mathrm{si}\,(Nx) - \frac{\pi}{2} + \mathcal{O}(N^{-1}) \quad (x \in (0,\pi)),$$

where $\mathrm{si}\,(y) := \int\limits_0^y \frac{\sin t}{t}\,\mathrm{d}t$. We are interested in the behavior of

$$(\mathcal{F}_N * g)\left(\frac{2\pi l}{N}\right) - g\left(\frac{2\pi l}{N}\right) = \mathrm{si}\,(2\pi l) - \frac{\pi}{2} + \mathcal{O}(N^{-1}) \quad (l = 0,\dots,\left\lceil\frac{N}{2}\right\rceil - 1).$$

Here $\lceil x \rceil$ denotes the smallest integer $\geq x$. It is well known that $\lim\limits_{x\to\infty} \mathrm{si}\,(x) = \frac{\pi}{2}$. Thus, if $l = l(N) \to \infty$ for $N \to \infty$, then, for any $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon)$ so that

$$\left|(\mathcal{F}_N * g)\left(\frac{2\pi l}{N}\right) - g\left(\frac{2\pi l}{N}\right)\right| < \frac{\pi h_-}{2\alpha_1}\,\varepsilon \quad \text{for all} \quad N \geq N_0. \tag{2.7}$$

p1a

The same holds if we approach 0 from the left, i.e. if we consider $2\pi l/N$ for $l = \lceil\frac{N}{2}\rceil, \dots, N-1$. Next we have by definition of $g$ and $h$ that

$$\tilde{h}(x) := h(x) - \frac{\alpha_1}{\pi}\,g(x)$$

is a continuous function. Since $\mathcal{F}_N$ is a reproducing kernel, for any $\varepsilon > 0$, there exists $\tilde{N}_0 = \tilde{N}_0(\varepsilon)$ so that for all $l \in \{0,\dots,N-1\}$

$$\left|(\mathcal{F}_N * \tilde{h})\left(\frac{2\pi l}{N}\right) - \tilde{h}\left(\frac{2\pi l}{N}\right)\right| < \frac{\varepsilon}{2}\,h_- \quad \text{for all} \quad N \geq \tilde{N}_0 \tag{2.8}$$

p2

Assume that $l = l(N) \to \infty$ for $N \to \infty$ ($l \in \{0, \ldots, \lceil \frac{N}{2} \rceil - 1\}$). Then we obtain by (2.7) and (2.8) that for any $\varepsilon > 0$ there exists $N(\varepsilon) = \max\,(N_0, \tilde{N}_0)$ so that

$$\begin{aligned}
\left|(\mathcal{F}_N * h)\left(\frac{2\pi l}{N}\right) - h\left(\frac{2\pi l}{N}\right)\right| &\le \left|(\mathcal{F}_N * \tilde{h})\left(\frac{2\pi l}{N}\right) - \tilde{h}\left(\frac{2\pi l}{N}\right)\right| \\
&+ \frac{\alpha_1}{\pi}\left|(\mathcal{F}_N * g)\left(\frac{2\pi l}{N}\right) - g\left(\frac{2\pi l}{N}\right)\right| \\
\left|(\mathcal{F}_N * h)\left(\frac{2\pi l}{N}\right) - h\left(\frac{2\pi l}{N}\right)\right| &\le \varepsilon\, h_- \qquad \text{for all} \quad N \ge N(\varepsilon)
\end{aligned}$$

and consequently, since $|h\left(\frac{2\pi l}{N}\right)| \ge h_- \quad (l \in I_N(f))$,

$$1 - \varepsilon \le \frac{\left|(\mathcal{F}_N * h)\left(\frac{2\pi l}{N}\right)\right|}{\left|h\left(\frac{2\pi l}{N}\right)\right|} \le 1 + \varepsilon \qquad (l \in I_N(f))\,. \tag{2.9}$$ p3

Let $m \le \mu + \nu$ denote the number of zeros of $f$ which are equal to one of the points $2\pi l/N$ $(l = 0, \ldots, N-1)$. Then the set

$$\left\{ \frac{\left|(\mathcal{F}_N * h)\left(\frac{2\pi l}{N}\right)\right|}{\left|h\left(\frac{2\pi l}{N}\right)\right|} : l \in I_N(f) \right\}$$

contains at least $N - m$ absolute values of eigenvalues of $\boldsymbol{M}_{N,f}(|h|)^{-1}\boldsymbol{M}_N(\mathcal{F}_N * h)$ and we conclude by (2.9) that except for $\mathcal{O}(1)$ eigenvalues and sufficently large $N$, all eigenvalues of $\boldsymbol{M}_{N,f}(|h|)^{-1}\boldsymbol{M}_N(\mathcal{F}_N * h)$ have absolute values contained in $[1-\varepsilon, 1+\varepsilon]$. This completes the proof. ■

R2.3 **Remark 2.3** In a similar way as above we can prove that for any $\varepsilon > 0$ and $N$ sufficiently large, the number of eigenvalues of $\boldsymbol{A}_N(h)$ with absolute values not in the interval $[h_- - \varepsilon, h_+]$ is $\mathcal{O}(\log N)$.

Note that the property that at most $o(N)$ eigenvalues of $\boldsymbol{A}_N(h)$ have absolute values not contained in $[h_- - \varepsilon, h_+]$ follows simply from the fact that the singular values of $\boldsymbol{A}_N(h)$ are distributed as $|h|$ [9, 15]. □

T2.3 **Theorem 2.4** *Let $f = p_s h \in \mathcal{L}_{2\pi}$ be given by (1.2) – (1.4). Then, for any $\varepsilon > 0$ and sufficiently large $N$, except for $\mathcal{O}(\log N)$ singular values, all singular values of*

$$\boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}}\boldsymbol{A}_N(f)\boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}}$$

*are contained in* $[1-\varepsilon, 1+\varepsilon]$.

**Proof.** The polynomial $p_s$ in (1.3) can be rewritten as

$$p_s = p\bar{p}\,,$$

where

$$p(t) := \prod_{j=1}^{\mu} (1 - \mathrm{e}^{-\mathrm{i}t_j}\,\mathrm{e}^{\mathrm{i}t})^{s_j}\,, \quad \sum_{j=1}^{\mu} s_j = s\,.$$

By straightforward computation it is easy to check that

$$\begin{aligned}
\boldsymbol{A}_N(f) &= \boldsymbol{A}_N(p_s\,h) \;=\; \boldsymbol{A}_N(p\,h\,\bar{p}) \\
&= \boldsymbol{A}_N(p\,h)\,\boldsymbol{A}_N(\bar{p}) \;+\; \boldsymbol{R}_N^c(s) \\
&= \boldsymbol{A}_N(p)\,\boldsymbol{A}_N(h)\,\boldsymbol{A}_N(\bar{p}) \;+\; \boldsymbol{R}_N^r(s)\,\boldsymbol{A}_N(\bar{p}) \;+\; \boldsymbol{R}_N^c(s) \\
&= \boldsymbol{A}_N(p)\,\boldsymbol{A}_N(h)\,\boldsymbol{A}_N(\bar{p}) \;+\; \boldsymbol{R}_N(2s)\,, \qquad (2.10)
\end{aligned}$$

b1

where only the first $s$ columns (rows) of $\boldsymbol{R}_N^{c(r)}(s)$ are nonzero columns (rows).
Since $|f| = p\,\bar{p}\,|h|$ the eigenvalues of $\boldsymbol{M}_{N,f}(|f|)^{-1}\boldsymbol{A}_N(f)$ coincide with the eigenvalues of

$$\boldsymbol{B}_N(f) := \boldsymbol{M}_{N,f}(|h|)^{-1/2}\,\boldsymbol{M}_{N,f}(p)^{-1}\,\boldsymbol{A}_N(f)\boldsymbol{M}_{N,f}(\bar{p})^{-1}\,\boldsymbol{M}_{N,f}(|h|)^{-1/2}\,. \qquad (2.11)$$

bnf

Now we obtain by (2.10), (2.1) and (2.3) that

$$\begin{aligned}
\boldsymbol{B}_N(f) &= \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{M}_{N,f}(p)^{-1}\boldsymbol{A}_N(p)\boldsymbol{A}_N(h)\boldsymbol{A}_N(\bar{p})\boldsymbol{M}_{N,f}(\bar{p})^{-1}\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}} + \boldsymbol{R}_N(2s) \\
&= \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{M}_{N,f}(p)^{-1}(\boldsymbol{M}_{N,f}(p) + \boldsymbol{R}_N(s+m))\boldsymbol{A}_N(h)\;\cdot \\
&\quad\; (\boldsymbol{M}_{N,f}(\bar{p}) + \boldsymbol{R}_N(s+m))\boldsymbol{M}_{N,f}(\bar{p})^{-1}\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}} + \boldsymbol{R}_N(2s) \\
&= \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{A}_N(h)\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}} \;+\; \boldsymbol{R}_N(4s+2m)\,. \qquad (2.12)
\end{aligned}$$

b2

By Lemma 2.2, for any $\varepsilon > 0$ and $N$ sufficiently large, except for $\mathcal{O}(\log N)$ singular values, all singular values of $\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{A}_N(h)\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}$ are contained in $[1-\varepsilon, 1+\varepsilon]$. Now the assertion follows by (2.12) and Weyl's interlacing theorem [8, p. 184]. ■

## 3 Circulant preconditioners involving positive kernels

In many applications we only know the entries $a_k(f)$ of the Toeplitz matrices $\boldsymbol{A}_N(f)$, but not the generating function itself. In this case, we use even positive reproducing kernels $K_N \in C_{2\pi}$. These are trigonometric polynomials of the form

$$K_N(t) := c_{N,0} + 2\sum_{k=1}^{N-1} c_{N,k}\cos kt, \quad c_{N,k} = a_k(K_N) \in \mathbb{R}$$

satisfying $K_N \geq 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,\mathrm{d}t = 1 \qquad (3.1)$$

k1

and the *reproducing property*

$$\lim_{N\to\infty} \|f - K_N * f\|_\infty = 0 \quad \text{for all} \quad f \in C_{2\pi}.$$

Since

$$(K_N * f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_N(x-t)\,\mathrm{d}t = \sum_{k=-(N-1)}^{N-1} a_k(f)\,c_{N,k}\,\mathrm{e}^{\mathrm{i}kx}\,,$$

the cyclic convolution of $K_N$ and $f$ is determined by the first $N$ Fourier coefficients of $f$. As preconditioner which can be constructed from the entries of $\boldsymbol{A}_N(f)$ without explicit knowledge of $f$ we suggest the circulant matrix $\boldsymbol{M}_{N,K_N*f}(|K_N * f|)$.

In order to obtain a suitable distribution of the eigenvalues of the preconditioned matrices, we need kernels with a special property which is related to the order

$$\sigma := \max_{j=1,\ldots,\mu} s_j$$

of the zeros of $p_s$.
The *generalized Jackson kernels* $\mathcal{J}_{m,N}$ of degree $\leq N-1$ are defined by

$$K_{m,N}(t) = \mathcal{J}_{m,N}(t) := \lambda_{m,N} \left( \frac{\sin(nt/2)}{\sin(t/2)} \right)^{2m} \qquad (m \in \mathbb{N}) \,, \tag{3.2}$$ j

where $n := \lfloor \frac{N-1}{m} \rfloor + 1$ and where $\lambda_{m,N}$ is determined by (3.1). Here $\lfloor t \rfloor$ denotes the largest integer $\leq t$. In particular, we have that

$$\lambda_{m,N} \sim N^{1-2m} \,,$$

i.e. there exist positive constants $c_1, c_2$ so that $c_1 \, N^{1-2m} \leq \lambda_{m,N} \leq c_2 \, N^{1-2m}$. See [6, p. 203 - 204]. A possibility for the construction of the Fourier coefficients of $\mathcal{J}_{m,N}$ is prescribed in [5].
The *B-spline kernels* $\mathcal{B}_{m,N}$ of degree $\leq N-1$ are defined by

$$K_{m,N}(t) = \mathcal{B}_{m,N}(t) := \frac{N}{m} \frac{1}{M_{2m}(0)} \sum_{r \in Z} \left( \operatorname{sinc} \left( \frac{N}{m} \left( \frac{t + 2\pi r}{2} \right) \right) \right)^{2m} \,, \tag{3.3}$$ b

where $M_m$ denotes the *centered cardinal B–spline* of order $m$ and

$$\operatorname{sinc} t := \begin{cases} \frac{\sin t}{t} & t \neq 0 \,, \\ 1 & t = 0 \,. \end{cases}$$

See [12, 4]. Since

$$\mathcal{B}_{m,N}(t) := 1 + \frac{2}{M_{2m}(0)} \sum_{k=1}^{N-1} M_{2m} \left( \frac{mk}{N} \right) \cos kt$$

the Fourier coefficients of $\mathcal{B}_{m,N}$ are given by values of centered cardinal $B$-splines. Note that $\mathcal{J}_{1,N} = \mathcal{B}_{1,N}$ is just the Fejér kernel $\mathcal{F}_N$.
The above kernels have the following important property:

T3.1 **Theorem 3.1** *Let $f = p_s h \in \mathcal{L}_{2\pi}$ be given by (1.2) – (1.4). Assume that for all $t_j$ ($j \in \{1, \ldots, \mu\}$) with $t_j = \xi_k$ for some $k \in \{1, \ldots, \nu\}$ and $\operatorname{sgn} h(\xi_k + 0) \neq \operatorname{sgn} h(\xi_k - 0)$ there exists a neighborhood $[t_j - \varepsilon_j, t_j + \varepsilon_j]$ ($\varepsilon_j > 0$) of $t_j$ so that $f$ is a monotone function in this neighborhood and moreover $f(t_j - t) = -f(t_j + t)$ ($0 \leq t \leq \varepsilon_j$). Let $K_N = K_{m,N}$ be given by (3.2) or (3.3), where*

$$m \geq \sigma + 1 \,.$$

*Then there exist $0 < \alpha \leq \beta < \infty$ so that for $N \to \infty$, except for $\mathcal{O}(1)$ points, all points of the set $\{2\pi l/N : l \in I_N(f)\}$ fulfill*

$$\frac{1}{\beta} \leq \frac{|(K_N * f)(\frac{2\pi l}{N})|}{|f(\frac{2\pi l}{N})|} \leq \frac{1}{\alpha} \,. \tag{3.4}$$ ke

**Proof.** 1. First we consider the upper bound. Since $p_s$ and $K_N$ are nonnegative, we obtain

$$\begin{aligned}
|(K_N * f)(x)| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(t)|\, p_s(t)\, K_N(x-t)\, \mathrm{d}t \\
&\leq h_+ \frac{1}{2\pi} \int_{-\pi}^{\pi} p_s(t) K_N(x-t)\, \mathrm{d}t \;=\; h_+\, (K_N * p_s)(x)\,.
\end{aligned}$$

In post99,chying99 [12, 5], we proved that $m \geq \sigma + 1$ implies that for all $x \in I_N(p_s) \supseteq I_N(f)$, there exists a constant $0 < c < \infty$ so that

$$\frac{(K_N * p_s)(x)}{p_s(x)} \leq c\,.$$

Thus, since $|h(x)| \geq h_-$ for $(x \in I_N(f))$, we obtain

$$\frac{|(K_N * f)(x)|}{|f(x)|} \leq \frac{h_+}{h_-}\, \frac{(K_N * p_s)(x)}{p_s(x)} \leq \frac{h_+}{h_-}\, c \qquad (x \in I_N(f))\,.$$

2. Next we deal with the lower bound.
2.1. Let $x \in I_N(f)$ be not in the neighborhood of $t_j$ $(j = 1, \ldots, \mu)$, i.e. there exist $b_j > 0$ independent of $N$ so that $|x - t_j| \geq b_j > 0$ $(j = 1, \ldots, \mu)$. Then $|f(x)| \geq c > 0$ for all $x \in I_N(f)$. Further, since $K_N$ is a reproducing kernel and by using the same arguments as in the proof of Lemma L2.2 2.2 if $x$ is in the neighborhood of some $\xi_k$ $(k = 1, \ldots, \nu)$, we obtain that, for any $\varepsilon > 0$ there exists $N(\varepsilon)$, so that except for at most a constant number of points, all considered points $x \in I_N(f)$ satisfy

$$|(K_N * f)(x) \,-\, f(x)| \leq c\,\varepsilon \qquad (N \geq N(\varepsilon))$$

and thus

$$\frac{|(K_N * f)(x)|}{|f(x)|} \geq 1 - \frac{c\,\varepsilon}{|f(x)|} \geq 1 - \varepsilon\,.$$

2.2. It remains to consider the points $x = x(N) \in I_N(f)$ with $\lim\limits_{N\to\infty} x(N) = t_j$ $(j = 1, \ldots, \mu)$. For simplicity we assume that

$$p_s(t) = (2 - 2\cos t)^s = (2\,\sin(t/2))^{2s}\,,$$

i.e. $p_s$ has only a zero of order $2s$ in $t_1 = 0$. Let $x = x(N) \in I_N(f)$ with

$$\lim_{N\to\infty} x(N) = 0\,.$$

For any fixed $0 < b < \pi$ we obtain

$$\begin{aligned}
(K_N * f)(x) &= \frac{1}{2\pi}\,\Big(\int_{-b}^{b} f(t) K_N(x-t)\, \mathrm{d}t \,+ \int_{-\pi}^{-b} f(t) K_N(x-t)\, \mathrm{d}t \,+ \int_{b}^{\pi} f(t) K_N(x-t)\, \mathrm{d}t\,\Big) \\
&= \frac{1}{2\pi}\,\Big(\int_{-b}^{b} f(t) K_N(x-t)\, \mathrm{d}t \,+ \int_{b+x}^{\pi+x} f(x-t) K_N(t)\, \mathrm{d}t \,+ \int_{b-x}^{\pi-x} f(x+t) K_N(t)\, \mathrm{d}t\,\Big)
\end{aligned}$$

and since $f$ is bounded

$$(K_N * f)(x) - \frac{1}{2\pi} \int\limits_{-b}^{b} f(t) K_N(x-t)\,\mathrm{d}t \;\sim\; \Big( \int\limits_{b+x}^{\pi+x} + \int\limits_{b-x}^{\pi-x} \Big) K_N(t)\,\mathrm{d}t\,.$$

By definition of $K_N$ we see that for any fixed $0 < \tilde{b} \leq \pi$

$$\int\limits_{\tilde{b}}^{\pi} K_N(t)\,\mathrm{d}t \;\leq\; \mathrm{const}\; N^{-2m+1}\,, \qquad (3.5) \quad \boxed{\texttt{t0}}$$

so that we get for small $x$ (e.g. $x < b/2$)

$$(K_N * f)(x) = \frac{1}{2\pi} \int\limits_{-b}^{b} f(t) K_N(x-t)\,\mathrm{d}t \;+\; \mathcal{O}(N^{-2m+1})\,. \qquad (3.6) \quad \boxed{\texttt{t1}}$$

2.2.1. Assume that $h$ has no jump in $t_1 = 0$ with sign change. Then there exists $\varepsilon > 0$ so that $h \geq h_-$ or $h \leq -h_-$ in $[-\varepsilon, \varepsilon]$. We restrict our attention to the case $h \geq h_-$. Since $0 < h_-\, p_s(t) \leq f(t) \leq h_+\, p_s(t)$ $(t \in [-\varepsilon, \varepsilon])$ and $p_s$ is monotone increasing on $(0, \pi)$, we obtain for $x(N) \in (0, \varepsilon) \cap I_N(f)$ and $N$ sufficiently large that

$$\begin{aligned}
\int\limits_{-\varepsilon}^{\varepsilon} \frac{f(t)}{f(x(N))} K_N(t - x(N))\,\mathrm{d}t \;&\geq\; \int\limits_{x(N)}^{\varepsilon} \frac{f(t)}{f(x(N))} K_N(t - x(N))\,\mathrm{d}t \\
&\geq\; \frac{h_-}{h_+} \int\limits_{x(N)}^{\varepsilon} \frac{p_s(t)}{p_s(x(N))} K_N(t - x(N))\,\mathrm{d}t \\
&\geq\; \frac{h_-}{h_+} \int\limits_{0}^{\varepsilon - x(N)} \frac{p_s(t)}{p_s(x(N))} K_N(t)\,\mathrm{d}t \;\geq\; c
\end{aligned} \qquad (3.7) \quad \boxed{\texttt{t11}}$$

with a positive constant $c$ independent of $N$. On the other hand, we have by definition of $p_s$ and since by assumption $s \leq m - 1$ that $f(x(N)) \geq h_-\, \tilde{c}\, N^{-2s} \geq h_-\, \tilde{c}\, N^{-2m+2}$. Then we obtain by (3.6) with $b = \varepsilon$ and (3.7) that for $N$ large enough

$$\frac{(K_N * f)(x(N))}{f(x(N))} \geq \mathrm{const}$$

with a positive constant const independent of $N$.
The proof for $x(N) \in (-\varepsilon, 0) \cap I_N(f)$ follows the same lines.

2.2.2. Finally, we assume that $h$ has a jump in $t_1 = 0$ with $\operatorname{sgn} h(0+0) \neq \operatorname{sgn} h(0-0)$. Without loss of generality let $h(0+0) > 0$. Then, by assumption on $f$, there exists $\varepsilon_1 > 0$ so that $h(t) = -h(-t)$ for $t \in [0, \varepsilon_1]$. Thus,

$$\int\limits_{-\varepsilon_1}^{\varepsilon_1} f(t) K_N(x-t)\,\mathrm{d}t = \int\limits_{0}^{\varepsilon_1} f(t)\big(K_N(t-x) - K_N(t+x)\big)\,\mathrm{d}t\,. \qquad (3.8) \quad \boxed{\texttt{t2}}$$

We consider points of the form

$$y = y_k(N) := \frac{2\pi m}{N\,\gamma}\,k \qquad (k \in \mathbb{N})$$

with $\lim\limits_{N\to\infty} y_k(N) = 0$, where $\gamma := mn/N$ in case of Jackson kernels and $\gamma := 1$ in case of $B$-spline kernels. Then we have for $t \in [0, \varepsilon_1]$ that

$$\mathcal{J}_{m,N}(t-y) - \mathcal{J}_{m,N}(t+y) = \lambda_{m,N}\left(\left(\frac{\sin(nt/2)}{\sin((t-y)/2)}\right)^{2m} - \left(\frac{\sin(nt/2)}{\sin((t+y)/2)}\right)^{2m}\right) \tag{3.9}$$

jpos

and consequently for sufficiently small $\varepsilon_1$ and $y$, since sin is odd and monotone increasing on $(0, \pi/2)$ that

$$\mathcal{J}_{m,N}(t-y) - \mathcal{J}_{m,N}(t+y) > 0 \quad \text{for all } t \in (0, \varepsilon_1)\,.$$

Further, by definition of the $B$-spline kernels

$$\mathcal{B}_{m,N}(t-y) - \mathcal{B}_{m,N}(t+y) = \mathcal{B}^0_{m,N}(t-y) - \mathcal{B}^0_{m,N}(t+y) \;+\; \mathcal{O}(N^{-2m+1})\,,$$

where $\mathcal{B}^0_{m,N}(t) := \frac{N}{m}\,\frac{1}{M_{2m}(0)}\,\left(\operatorname{sinc}\left(\frac{N}{m}\frac{t}{2}\right)\right)^{2m}$ and similarly as in (3.9) we see that

$$\mathcal{B}^0_{m,N}(t-y) - \mathcal{B}^0_{m,N}(t+y) > 0 \quad \text{for all } t \in (0, \varepsilon_1)\,.$$

By assumption $h$ does not change the sign in $(0, \varepsilon_1)$. Then we obtain by (3.8), monotonicity of $p_s$ in $(0, \pi)$ and $m \geq s+1$ that

$$\int\limits_{-\varepsilon_1}^{\varepsilon_1} \frac{f(t)}{f(y)} K_N(y-t)\,\mathrm{d}t \geq \frac{h_-}{h_+}\int\limits_{y}^{\varepsilon_1} K^0_N(t-y) - K^0_N(t+y)\,\mathrm{d}t \;+\; \mathcal{O}(N^{-1})\,, \tag{3.10}$$

t3

where $K^0_N \in \{\mathcal{J}_{m,N}, \mathcal{B}^0_{m,N}\}$. Set $w = w(N) := \frac{2\pi m}{N\gamma}$. Then $y_k = y_k(N) = w\,k$ and there exist $r = r(N) \in \mathbb{N}$ $(r > k)$ so that $\varepsilon_1 = w\,r + \tilde{\varepsilon}_1$, where $0 \leq \tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(N) < w$. Now it follows

$$\begin{aligned}
\int\limits_{y_k}^{wr} K^0_N(t-y_k) - K^0_N(t+y_k)\,\mathrm{d}t &= \sum_{l=0}^{r-k-1}\;\int\limits_{y_k+wl}^{y_k+w(l+1)} K^0_N(t-y_k) - K^0_N(t+y_k)\,\mathrm{d}t \\
&= \sum_{l=0}^{2k-1}\;\int\limits_{wl}^{w(l+1)} K^0_N(t)\,\mathrm{d}t \;-\; \sum_{l=r-k}^{r+k-1}\;\int\limits_{wl}^{w(l+1)} K^0_N(t)\,\mathrm{d}t \\
&\geq \int\limits_{0}^{w} K^0_N(t)\,\mathrm{d}t \;-\; \int\limits_{\varepsilon_1+y_k-w}^{\varepsilon_1+y_k} K^0_N(t)\,\mathrm{d}t
\end{aligned}$$

and further by (3.5) and since $\lim\limits_{N\to\infty} y_k = 0$,

$$\int\limits_{y_k}^{\varepsilon_1} K^0_N(t-y_k) - K^0_N(t+y_k)\,\mathrm{d}t \geq \int\limits_{0}^{w} K^0_N(t)\,\mathrm{d}t \;+\; \mathcal{O}(N^{-2m+1})\,.$$

Straightforward computation yields

$$\int_0^{2\pi m/(N\gamma)} K_N^0(t)\,\mathrm{d}t \;\geq\; \text{const} \int_0^{\pi} \left(\frac{\sin u}{u}\right)^{2m} \mathrm{d}u \;\geq\; \text{const}\,.$$

Hence we get for $N$ large enough that

$$\int_{y_k}^{\varepsilon_1} K_N^0(t-y_k) - K_N^0(t+y_k)\,\mathrm{d}t \;\geq\; \text{const}$$

and by (t3 3.10) that

$$\int_{-\varepsilon_1}^{\varepsilon_1} \frac{f(t)}{f(y_k)} K_N(y_k - t)\,\mathrm{d}t \;\geq\; \text{const} \tag{3.11}$$

t4

with positive constants const independent of $N$.
Now we consider $x(N) \in I_N(f)$ with $y_k(N) \leq x(N) < y_{k+1}(N)$.
Let $z(N) := x(N) - y_k(N) > 0$. Then

$$\begin{aligned}
\int_{-\varepsilon_1}^{\varepsilon_1} f(t)\,K_N(t-x(N))\,\mathrm{d}t &= \int_{-\varepsilon_1 - z(N)}^{\varepsilon_1 - z(N)} f(t+z(N))\,K_N(t-y_k(N))\,\mathrm{d}t \\
&= \int_{-\varepsilon_1}^{\varepsilon_1 - z(N)} f(t+z(N))\,K_N(t-y_k(N))\,\mathrm{d}t \\
&\quad + \int_{-\varepsilon_1 - z(N)}^{-\varepsilon_1} f(t+z(N))\,K_N(t-y_k(N))\,\mathrm{d}t
\end{aligned}$$

and since $f$ is by assumption monotone increasing on $[-\varepsilon_1, \varepsilon_1]$

$$\begin{aligned}
\int_{-\varepsilon_1}^{\varepsilon_1} f(t)\,K_N(t-x(N))\,\mathrm{d}t &\geq \int_{-\varepsilon_1}^{\varepsilon_1 - z(N)} f(t)\,K_N(t-y_k(N))\,\mathrm{d}t \;+\; \int_{-\varepsilon_1}^{-\varepsilon_1 + z(N)} f(t)\,K_N(t-x(N))\,\mathrm{d}t \\
&= \int_{-\varepsilon_1}^{\varepsilon_1} f(t)\,K_N(t-y_k(N))\,\mathrm{d}t \;+\; \int_{-\varepsilon_1}^{-\varepsilon_1 + z(N)} f(t)\,K_N(t-x(N))\,\mathrm{d}t \\
&\quad - \int_{\varepsilon_1 - z(N)}^{\varepsilon_1} f(t)\,K_N(t-y_k(N))\,\mathrm{d}t
\end{aligned}$$

and by (t0 3.5) and since $f$ is bounded

$$\int_{-\varepsilon_1}^{\varepsilon_1} f(t)\,K_N(t-x(N))\,\mathrm{d}t \;\geq\; \int_{-\varepsilon_1}^{\varepsilon_1} f(t)\,K_N(t-y_k(N))\,\mathrm{d}t \;+\; \mathcal{O}(N^{-2m+1})\,. \tag{3.12}$$

t5

By assumption $x(N) = \zeta\, y_k(N)$ $(0 < \zeta < 2)$. Thus

$$\frac{\int_{-\varepsilon_1}^{\varepsilon_1} f(t)\, K_N(t - x(N))\, \mathrm{d}t}{f(x(N))} \;\geq\; \mathrm{const}\, \frac{\int_{-\varepsilon_1}^{\varepsilon_1} f(t)\, K_N(t - y_k(N))\, \mathrm{d}t}{f(y_k(N))}$$

and since $f(y_k(N) \geq \mathrm{const}\, N^{-2s}$ and $m \geq s + 1$ we obtain by (3.12), (3.11) that for $N$ large enough

$$\int_{-\varepsilon_1}^{\varepsilon_1} f(t)\, K_N(t - x(N))\, \mathrm{d}t \,/\, f(x(N)) \;\geq\; \mathrm{const}$$

with a nonnegative constant const independent of $N$. Finally, we use (3.6) with $b = \varepsilon_1$ and again $m \geq s + 1$ to finish the proof. ■

To show our main result we also need the following lemma.

L3.2 **Lemma 3.2** *Let $\boldsymbol{A} \in \mathbb{C}^{N,N}$ be a Hermitian positive definite matrix having $N - n_1$ eigenvalues in $[a_-, a_+]$, where $0 < a_- \leq a_+ < \infty$. Let $\boldsymbol{B} \in \mathbb{C}^{N,N}$ be a Hermitian matrix with $N - n_2$ singular values in $[b_-, b_+]$, where $0 < b_- \leq b_+ < \infty$. Then at least $N - 4n_1 - n_2$ eigenvalues of $\boldsymbol{A}\, \boldsymbol{B}$ are contained in $[-a_+b_+, -a_-b_-] \cup [a_-b_-, a_+b_+]$.*

**Proof.** 1. Assume first that $n_1 = 0$, i.e. $\boldsymbol{A}$ has only eigenvalues in $[a_-, a_+]$. Let $\lambda_j(\boldsymbol{B})$ denote the $j$–th eigenvalue of the matrix $\boldsymbol{B}$. We consider the eigenvalues of $\boldsymbol{B} - t\boldsymbol{A}^{-1}$ with respect to $t \in \boldsymbol{R}$. By Weyl's interlacing theorem (see [8, p. 184]) we obtain for $t \geq 0$ that

$$\lambda_j(\boldsymbol{B}) - \frac{t}{a_-} \leq \lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) \leq \lambda_j(\boldsymbol{B}) - \frac{t}{a_+} \qquad (3.13)$$
g1

and for $t < 0$ that

$$\lambda_j(\boldsymbol{B}) - \frac{t}{a_+} \leq \lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) \leq \lambda_j(\boldsymbol{B}) - \frac{t}{a_-}\,. \qquad (3.14)$$
g2

Let $\lambda_j(\boldsymbol{B}) \in [-b_+, -b_-]$. Then we obtain by (3.13) and (3.14) that $\lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) < 0$ for all $t > -a_-b_-$. On the other hand, we see by (3.13) and (3.14) that $\lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) > 0$ for all $t < -a_+b_+$. Thus, since $\lambda_j\left(\boldsymbol{B} - t\boldsymbol{A}^{-1}\right) = \lambda_j(t)$ is a continuous function in $t \in \boldsymbol{R}$, there exists $t_j \in [-a_+b_+, -a_-b_-]$ such that $\lambda_j\left(\boldsymbol{B} - t_j\boldsymbol{A}^{-1}\right) = 0$. This implies that $t_j \in [-a_+b_+, -a_-b_-]$ is an eigenvalue of $\boldsymbol{AB}$. Consequently, every $\lambda_j(\boldsymbol{B}) \in [-b_+, -b_-]$ corresponds to an eigenvalue $t_j \in [-a_+b_+, -a_-b_-]$ of $\boldsymbol{AB}$. (Eigenvalues are called with multiplicities.)
The examination of $\lambda_j(\boldsymbol{B}) \in [a_-b_-, a_+b_+]$ follows the same lines.
In summary, $N - n_2$ eigenvalues of $\boldsymbol{AB}$ are contained in $[-a_+b_+, -a_-b_-] \cup [a_-b_-, a_+b_+]$.
2. Let $n_1$ eigenvalues of $\boldsymbol{A}$ be outside $[a_-, a_+]$. Then, since $\boldsymbol{A}$ is positive definite, the matrix can be splitted as

$$\boldsymbol{A}^{1/2} = \tilde{\boldsymbol{A}}^{1/2} + \boldsymbol{R}(n_1)\,, \qquad (3.15)$$
g3

where $\tilde{\boldsymbol{A}}^{1/2}$ is Hermitian with all eigenvalues in $[a_-^{1/2}, a_+^{1/2}]$ and $\boldsymbol{R}(n_1)$ is a Hermitian matrix of rank $n_1$. The eigenvalues of $\boldsymbol{AB}$ coincide with the eigenvalues of $\boldsymbol{A}^{1/2}\boldsymbol{B}\boldsymbol{A}^{1/2}$. Hence it remains to show that at most $4n_1 + n_2$ singular values of $\boldsymbol{A}^{1/2}\boldsymbol{B}\boldsymbol{A}^{1/2}$ are not contained in $[a_-b_-, a_+b_+]$. By (3.15) we have

$$\begin{aligned} \boldsymbol{A}^{1/2}\boldsymbol{B}\boldsymbol{A}^{1/2} &= \tilde{\boldsymbol{A}}^{1/2}\boldsymbol{B}\tilde{\boldsymbol{A}}^{1/2} + \boldsymbol{R}(2n_1)\,, \\ \left(\boldsymbol{A}^{1/2}\boldsymbol{B}\boldsymbol{A}^{1/2}\right)^2 &= \left(\tilde{\boldsymbol{A}}^{1/2}\boldsymbol{B}\tilde{\boldsymbol{A}}^{1/2}\right)^2 + \boldsymbol{R}(4n_1)\,. \end{aligned} \qquad (3.16)$$
g4

By 1. all but $n_2$ singular values of $\tilde{\boldsymbol{A}}^{1/2}\boldsymbol{B}\tilde{\boldsymbol{A}}^{1/2}$ are contained in $[a_-b_-, a_+b_+]$. Then (3.16) and Weyl's interlacing theorem yield the assertion. ■

**Theorem 3.3** *Let $f = p_s h \in \mathcal{L}_{2\pi}$ be given by (1.2) – (1.4). Assume that for all $t_j$ ($j \in \{1,\ldots,\mu\}$) with $t_j = \xi_k$ for some $k \in \{1,\ldots,\nu\}$ and $\operatorname{sgn} h(\xi_k + 0) \neq \operatorname{sgn} h(\xi_k - 0)$ there exists a neighborhood $[t_j - \varepsilon_j, t_j + \varepsilon_j]$ ($\varepsilon_j > 0$) of $t_j$ so that $f$ is a monotone function in this neighborhood and moreover $f(t_j - t) = -f(t_j + t)$ ($0 \le t \le \varepsilon_j$). Let $K_N = K_{m,N}$ be given by (3.2) or (3.3), where*

$$m \geq \sigma + 1\,.$$

*By $\alpha, \beta$ we denote the constants from Theorem 3.1.*
*Then, for any $\varepsilon > 0$ and sufficiently large $N$, except for $\mathcal{O}(\log N)$ singular values, all singular values of $\boldsymbol{M}_N(|K_N * f|)^{-\frac{1}{2}}\boldsymbol{A}_N(f)\boldsymbol{M}_N(|K_N * f|)^{-\frac{1}{2}}$ are contained in $[\alpha - \varepsilon, \beta + \varepsilon]$.*

**Proof.** Let $\boldsymbol{B}_N(f)$ be defined by (2.11). Then we obtain by (2.12) that

$$\begin{aligned}
& \boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-\frac{1}{2}}\, \boldsymbol{A}_N(f)\, \boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-\frac{1}{2}} && (3.17)\\
= \; & \boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-\frac{1}{2}}\, \boldsymbol{M}_{N,f}(p)\, \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}}\;\; \boldsymbol{B}_N(f)\;\cdot \\
& \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}}\, \boldsymbol{M}_{N,f}(\bar{p})\;\, \boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-\frac{1}{2}} \\
= \; & \boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-\frac{1}{2}}\, \boldsymbol{M}_{N,f}(p)\, \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}}\;\; \boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{A}_N(h)\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}} \\
& \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}}\, \boldsymbol{M}_{N,f}(\bar{p})\;\, \boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-\frac{1}{2}} \;+\; \boldsymbol{R}(4s + 2m)\,. && (3.18)
\end{aligned}$$

The distribution of the eigenvalues of $\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}\boldsymbol{A}_N(h)\boldsymbol{M}_{N,f}(|h|)^{-\frac{1}{2}}$ is known by Lemma 2.2. It remains to examine the eigenvalues of the Hermitian positive definite matrix

$$\boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}}\, \boldsymbol{M}_{N,f}(\bar{p})\, \boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-1}\, \boldsymbol{M}_{N,f}(p)\, \boldsymbol{M}_{N,f}(|h|)^{\frac{1}{2}}\,.$$

These eigenvalues coincide with the reciprocal eigenvalues of $\boldsymbol{M}_{N,f}(|f|)^{-1}\, \boldsymbol{M}_{N,K_N*f}(|K_N*f|)$. By definition of $\boldsymbol{M}_{N,g}$ and since $K_N$ is a reproducing kernel, except for $\mathcal{O}(1)$ eigenvalues, all eigenvalues of $\boldsymbol{M}_{N,f}(|f|)^{-1}\, \boldsymbol{M}_{N,K_N*f}(|K_N * f|)$ are given by $|(K_N * f)(2\pi l/N)|/|f(2\pi l/N)|$ ($l \in I_N(f)$). Thus, by Theorem 3.1, for $N \to \infty$ only $\mathcal{O}(1)$ eigenvalues of
$\boldsymbol{M}_{N,f}(|f|)\, \boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-1}$ are not contained in $[\alpha, \beta]$. Consequently, by (3.18), Lemma 2.2, Lemma 3.2 and Weyl's interlacing theorem at most $\mathcal{O}(\log N)$ singular values of of of $\boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-\frac{1}{2}}\boldsymbol{A}_N(f)\boldsymbol{M}_{N,K_N*f}(|K_N * f|)^{-\frac{1}{2}}$ are not contained in $[\alpha - \varepsilon, \beta + \varepsilon]$. ■

## 4 Trigonometric preconditioners

In addition to Section 2, we suppose that the Toeplitz matrices $\boldsymbol{A}_N \in \mathbb{R}^{N,N}$ are symmetric, i.e. the generating function $f \in \mathcal{L}_{2\pi}$ is even. This suggests the application of so-called trigonometric preconditioners. Note that in the symmetric case the multiplication of a vector with $\boldsymbol{A}_N$ can be realized using *fast trigonometric transforms* instead of fast Fourier transforms (see [10]). In this way complex arithmetic can be completely avoided in the iterative solution of (1.1). This is one of the reasons to look for preconditioners which can be diagonalized by trigonometric matrices corresponding to fast trigonometric transforms instead of the Fourier matrix $\boldsymbol{F}_N$.

In practice, four discrete sine transforms (DST I – IV) and four discrete cosine transforms (DCT I – IV) were used (see [17]). Any of these eight trigonometric transforms can be realized with $\mathcal{O}(N \log N)$ arithmetical operations. Likewise, we can define preconditioners with respect to any of these transforms.
In this paper, we restrict our attention to the so-called discrete cosine transform of type II (DCT-II) and discrete sine transform of type II (DST-II), which are determined by the following transform matrices:

$$\text{DCT–II} \quad : \quad \boldsymbol{C}_N^{II} := \left(\frac{2}{N}\right)^{1/2} \left(\epsilon_j^N \cos \frac{j(2k+1)\pi}{2N}\right)_{j,k=0}^{N-1} \in \mathbb{R}^{N,N},$$

$$\text{DST–II} \quad : \quad \boldsymbol{S}_N^{II} := \left(\frac{2}{N}\right)^{1/2} \left(\epsilon_{j+1}^N \sin \frac{(j+1)(2k+1)\pi}{2N}\right)_{j,k=0}^{N-1} \in \mathbb{R}^{N,N},$$

where $\epsilon_k^N := 2^{-1/2}(k = 0, N)$ and $\epsilon_k^N := 1$ $(k = 1, \ldots, N-1)$. We propose the preconditioners

$$\text{DCT} - \text{II}: \quad \boldsymbol{M}_{N,f}(|f|, \boldsymbol{C}_N^{II}) := (\boldsymbol{C}_N^{II})' \operatorname{diag}\left(|f(\tilde{x}_{N,l})|\right)_{l=0}^{N-1} \boldsymbol{C}_N^{II},$$

$$\text{DST} - \text{II}: \quad \boldsymbol{M}_{N,f}(|f|, \boldsymbol{S}_N^{II}) := (\boldsymbol{S}_N^{II})' \operatorname{diag}\left(|f(\tilde{x}_{N,l})|\right)_{l=1}^{N} \boldsymbol{S}_N^{II},$$

where

$$\tilde{x}_{N,l} := \begin{cases} \frac{l\pi}{N} & \text{if} \quad f\left(\frac{l\pi}{N}\right) \neq 0, \\ \frac{\tilde{l}\pi}{N} & \text{otherwise} \end{cases}$$

and where $\tilde{l} \in \{0, \ldots, N-1\}$ is the next higher index to $l$ such that $|f(\tilde{x}_{N,l})| > 0$. See [11]. Then we can prove in a completely similar way as in Section 2 that for any $\varepsilon > 0$ and sufficiently large $N$ except for $\mathcal{O}(\log N)$ singular values, all singular values of

$$\boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{-\frac{1}{2}} \boldsymbol{A}_N(f) \boldsymbol{M}_{N,f}(|f|, \boldsymbol{O})^{-\frac{1}{2}} \quad (\boldsymbol{O} \in \{\boldsymbol{S}_N^{II}, \boldsymbol{C}_N^{II}\})$$

are contained in $[1-\varepsilon, 1+\varepsilon]$.

## 5 Convergence of preconditioned MINRES

In order to prescribe the convergence behavior of preconditioned MINRES with our preconditioners of the previous sections, we have to estimate the smaller outlyers for increasing $N$.

L5.1 **Lemma 5.1** *Let $f \in \mathcal{L}_{2\pi}$ be defined by (1.2)–(1.4). Assume that $\kappa_2(\boldsymbol{A}_N(f)) = \mathcal{O}(N^\alpha)$ $(\alpha > 0)$. Then the smallest absolute values of the eigenvalues of $\boldsymbol{M}_{N,f}(|f|)^{-1} \boldsymbol{A}_N(f)$ and $\boldsymbol{M}_{N,K_N * f}(|K_N * f|)^{-1} \boldsymbol{A}_N(f)$ behave for $N \to \infty$ as $\mathcal{O}(N^{-\alpha})$.*

**Proof.** Since

$$\|\boldsymbol{A}_N(f)^{-1} \boldsymbol{M}_{N,f}(|f|)\|_2 \leq \frac{\|\boldsymbol{M}_{N,f}(|f|)\|_2}{\|\boldsymbol{A}_N(f)\|_2} \kappa_2(\boldsymbol{A}_N(f)),$$

$$\|\boldsymbol{A}_N(f)^{-1} \boldsymbol{M}_{N,K_N * f}(|K_N * f|)\|_2 \leq \frac{\|\boldsymbol{M}_{N,K_N * f}(|K_N * f|)\|_2}{\|\boldsymbol{A}_N(f)\|_2} \kappa_2(\boldsymbol{A}_N(f))$$

and both $\|\boldsymbol{M}_{N,f}(|f|)\|_2$ and $\|\boldsymbol{M}_{N,K_N * f}(|K_N * f|)\|_2$ are restricted from above, it remains to show that there exists a constant $c > 0$ independent of $N$ so that

$$\|\boldsymbol{A}_N(f)\|_2 > c.$$

The above inequality follows immediately from the fact that the singular values of $\boldsymbol{A}_N(f)$ are distributed as $|f|$ (see pa86,ty96 [9, 15]). ■

We want to combine our knowledge of the distribution of the eigenvalues of our preconditioned matrices with results concerning the convergence of MINRES.

T3.7 **Theorem 5.2** *Let $\boldsymbol{A} \in \mathbb{C}^{N,N}$ be a Hermitian matrix with $p$ and $q$ isolated large and small singular values, respectively:*

$$0 < \sigma_1 \le \sigma_2 \le \ldots \le \sigma_q \quad < \quad a \le \sigma_{q+1} \le \ldots \sigma_{N-p} \le b$$
$$< \quad \sigma_{N-p+1} \le \sigma_{N-p+2} \le \ldots \le \sigma_N \quad (0 < a \le b < \infty).$$

*Let $\nu(k) := 0$ if $k - p - q \equiv 0 \bmod 2$ and $\nu(k) := 1$ otherwise. Then MINRES requires for the solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$*

$$k \;\le\; 2\left(\ln\frac{2}{\tau} + \sum_{k=1}^{q}\ln\left(1 + \frac{b}{\sigma_k}\right) + p\ln 2\right) \;/\; \left(\ln\frac{1+(\frac{a}{b})}{1-(\frac{a}{b})}\right) + p + q + \nu(k)$$

*iteration steps to achieve precision $\tau$, i.e. $\frac{||\boldsymbol{r}^{(k)}||_2}{||\boldsymbol{r}^{(0)}||_2} \le \tau$ , where $\boldsymbol{r}^{(k)} := \boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}^{(k)}$ and $x^{(k)}$ is the $k$-th iterate.*

The theorem can be proved by using the same technique as in ax [1, pp. 569 - 573]. Namely, based on the known estimate

$$\frac{||\boldsymbol{r}^{(k)}||_2}{||\boldsymbol{r}^{(0)}||_2} \le \min_{p_k \in \Pi_k^0} \max_{\lambda_j} |p_k(\lambda_j)|\,,$$

where $\Pi_k^0$ denotes the space of polynomials of degree $\le k$ with $p_k(0) = 1$ and $\lambda_j$ are the eigenvalues of $\boldsymbol{A}$, we choose $p_k$ as product of the linear polynomials passing through the p+q outlyers and the modified Chebyshev polynomials

$$T_{\lfloor (k-p-q)/2 \rfloor}\left(1 + 2\,\frac{a^2 - x^2}{b^2 - a^2}\right) \,/\, T_{\lfloor (k-p-q)/2 \rfloor}\left(1 + 2\,\frac{a^2}{b^2 - a^2}\right).$$

The above summand $p \ln 2$ can be further reduced if we use polynomials of higher degree for the larger outlyers.

Note that a similar estimate can be given for the CG method applied to the normal equation $\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{b}$. Here we need

$$k \;\le\; \left(\ln\frac{2}{\tau} + \sum_{k=1}^{q}\ln\left(\frac{b}{\sigma_k^2}\right)\right) \;/\; \left(\ln\frac{1+(\frac{a}{b})}{1-(\frac{a}{b})}\right) + p + q$$

iteration steps to archive precision $\frac{\|\boldsymbol{e}^{(k)}\|_{\boldsymbol{A}}}{||\boldsymbol{e}^{(0)}||_{\boldsymbol{A}}} \le \tau$, where $\boldsymbol{e}^{(k)} := \boldsymbol{x}_* - \boldsymbol{x}^{(k)}$. Note that the latter method requires two matrix-vector multiplications in each iteration step.

By Theorem T2.3 2.4, Theorem T3.3 3.3 and Lemma L5.1 5.1 our preconditioned MINRES with both preconditioners $\boldsymbol{M}_{N,f}(|f|)$ and $\boldsymbol{M}_{N,K_N * f}(|K_N * f|)$ produces a solution of (1.1 1.1) of prescribed precision in $\mathcal{O}(\log N)$ iteration steps and with $\mathcal{O}(N \log^2 N)$ arithmetical operations. The same holds for preconditioned CG applied to the normal equation.

## 6 Numerical results

In this section, we test our circulant and trigonometric preconditioners in connection with different iterative methods on a SGI O2 work station. As transform length we use $N = 2^n$, as right-hand side $\boldsymbol{b}$ of (1.1) the vector consisting of $N$ entries "1" and as start vector the zero vector.
We begin with a comparision of MINRES applied to

$$\boldsymbol{M}_{N,f}(|f|,\boldsymbol{O})^{-1}\,\boldsymbol{A}_N(f)\,\boldsymbol{x} = \boldsymbol{M}_{N,f}(|f|,\boldsymbol{O})^{-1}\,\boldsymbol{b}\,, \tag{6.1}$$

where $\boldsymbol{O} \in \{\boldsymbol{F}_N,\, \boldsymbol{C}_N^{II},\, \boldsymbol{S}_N^{II}\}$ and CGNE (Craig's method) (cf. [13, p. 239]) applied to

$$(\boldsymbol{M}_{N,f}(|f|,\boldsymbol{O})^{-\frac{1}{2}}\,\boldsymbol{A}_N(f)\,\boldsymbol{M}_{N,f}(|f|,\boldsymbol{O})^{-\frac{1}{2}})\,(\boldsymbol{M}_{N,f}(|f|,\boldsymbol{O})^{\frac{1}{2}}\,\boldsymbol{x}) = \boldsymbol{M}_{N,f}(|f|,\boldsymbol{O})^{-\frac{1}{2}}\,\boldsymbol{b}\,. \tag{6.2}$$

For both algorithms we have used MATLAB implementations of B. Fischer. See also [7]. In particular, his implementation of preconditioned MINRES avoids the splitting (6.2).
At the end of this section, we will see that it is very important to use appropriate stopping criteria for the algorithms to obtain indeed a solution of the original system with a small residual $\boldsymbol{b} - \boldsymbol{A}_N\boldsymbol{x}^{(k)}$. In order to make the following computations with MINRES and CGNE comparable, we have stopped both computations if

$$||\boldsymbol{b} - \boldsymbol{A}_N\boldsymbol{x}^{(k)}||_2\,/\,||\boldsymbol{b}||_2 < 10^{-7}\,.$$

Note that the implementations by B. Fischer use other stopping rules. See iii) below.

**Example 1.** We begin with Hermitian Toeplitz matrices $\boldsymbol{A}_N(f)$ arising from the generating function

$$f_1(x) = h_1(x)\;x^2 \quad \text{with } h_1(x) = (x^2+1)\;\mathrm{sgn}\;(x) \quad (x \in [-\pi,\pi))\,.$$

Table 1 presents the number of iterations for circulant preconditioners. The first row of the table contains the exponent $n$ of the transform length $N = 2^n$. According to Theorem 2.4 and Theorem 5.2, the preconditioners $\boldsymbol{M}_N(|f|,\boldsymbol{F}_N)$ lead to very good results. As expected, the preconditioners $\boldsymbol{M}_{N,K_N*f}(|K_N*f|,\boldsymbol{F}_N)$ with the Fejér kernels $K_N = \mathcal{F}_N$ are not suitable for (1.1) (cf. also [12]), while the preconditioners with $K_N = \mathcal{B}_{2,N}$ do their job.
Further, CGNE needs half the number of iterations but twice the number of matrix-vector multiplications per iteration than MINRES. See also Section 5.
**Example 2.** Next, we consider the symmetric Toeplitz matrices $\boldsymbol{A}_N(f)$ arising from the generating function

$$f_2(x) = h_2(x)\,(\cos(x+2)+1)\,(\cos(x-2)+1)$$

with

$$h_2(x) = \mathrm{sgn}(x-\pi+2)\,\mathrm{sgn}(x+\pi-2)\,.$$

Tables 2 presents the number of iterations for trigonometric preconditioners. The results are similar to those of Example 1, except that CGNE requires nearly the same number of iterations as MINRES.

At the end of this section, we want to emphazise the influence of different stopping rules on the numerical solution of the system. We deal with Toeplitz systems (1.1), where $\boldsymbol{A}_N(f) =$

| method | $\boldsymbol{M}_{N,f}$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| MINRES | $\boldsymbol{I}_N$ | 23 | 71 | 277 | * | * | * | * |
| MINRES | $\boldsymbol{M}_{N,f}(\|f\|, \boldsymbol{F}_N)$ | 15 | 17 | 17 | 19 | 21 | 23 | 23 |
| MINRES | $\boldsymbol{M}_{N,\mathcal{F}_N * f}(\|\mathcal{F}_N * f\|, \boldsymbol{F}_N)$ | 19 | 31 | 35 | 41 | 43 | 47 | 51 |
| MINRES | $\boldsymbol{M}_{N,\mathcal{B}_{2,N} * f}(\|\mathcal{B}_{2,N} * f\|, \boldsymbol{F}_N)$ | 19 | 23 | 23 | 25 | 25 | 27 | 29 |
| CGNE | $\boldsymbol{I}_N$ | 11 | 37 | 164 | * | * | * | * |
| CGNE | $\boldsymbol{M}_{N,f}(\|f\|, \boldsymbol{F}_N)$ | 8 | 8 | 9 | 9 | 9 | 10 | 10 |

Table 1: $f(t) = h_1(t)\, t^2 \quad h_1(t) = (t^2+1)\ \mathrm{sgn}\,(t) \quad (t \in [-\pi, \pi))$

| method | $\boldsymbol{M}_N$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| MINRES | $\boldsymbol{I}_N$ | 9 | 17 | 45 | 142 | 401 | * | * |
| MINRES | $\boldsymbol{M}_{N,f}(\|f\|, \boldsymbol{C}_N^{II})$ | 8 | 9 | 10 | 11 | 14 | 13 | 16 |
| MINRES | $\boldsymbol{M}_{N,f}(\|f\|, \boldsymbol{S}_N^{II})$ | 9 | 10 | 11 | 12 | 14 | 13 | 16 |
| MINRES | $\boldsymbol{M}_{N,\mathcal{F}_N * f}(\|\mathcal{F}_N * f\|, \boldsymbol{C}_N^{II})$ | 10 | 15 | 20 | 26 | 30 | 39 | 53 |
| MINRES | $\boldsymbol{M}_{N,\mathcal{F}_N * f}(\|\mathcal{F}_N * f\|, \boldsymbol{S}_N^{II})$ | 10 | 15 | 19 | 25 | 30 | 39 | 53 |
| MINRES | $\boldsymbol{M}_{N,\mathcal{B}_{2,N} * f}(\|\mathcal{B}_{2,N} * f\|, \boldsymbol{C}_N^{II})$ | 9 | 15 | 17 | 16 | 20 | 18 | 18 |
| MINRES | $\boldsymbol{M}_{N,\mathcal{B}_{2,N} * f}(\|\mathcal{B}_{2,N} * f\|, \boldsymbol{S}_N^{II})$ | 9 | 14 | 16 | 18 | 19 | 18 | 18 |
| CGNE | $\boldsymbol{I}_N$ | 10 | 29 | 99 | 413 | * | * | * |
| CGNE | $\boldsymbol{M}_{N,f}(\|f\|^2, \boldsymbol{C}_N^{II})$ | 7 | 9 | 11 | 11 | 17 | 16 | 17 |
| CGNE | $\boldsymbol{M}_{N,f}(\|f\|^2, \boldsymbol{S}_N^{II})$ | 7 | 7 | 10 | 10 | 12 | 14 | 15 |

Table 2: $f_2(t) = h_2(t)\ (\cos(x+2)+1)\,(\cos(x-2)+1) \quad (t \in [-\pi, \pi))$

$\boldsymbol{A}_{128}(f_1)$. Beyond the above considered MINRES and CGNE with $\boldsymbol{O} = \boldsymbol{F}_N$, we also examine the CG method applied to

$$(\boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{A}_N(f))^*\ (\boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{A}_N(f))\boldsymbol{x} = (\boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{A}_N(f))^*\ \boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{b} \tag{6.3}$$

n2

and the preconditioned CG method (PCG) with preconditioner $\boldsymbol{M}_{N,f}(|f|^2)$ applied to

$$\boldsymbol{A}_N^*(f)\boldsymbol{A}_N(f)\boldsymbol{x} = \boldsymbol{A}_N^*(f)\boldsymbol{b}\,. \tag{6.4}$$

n1

For each of the above four algorithms, the following four figures compare
i) $\log_{10} \|\boldsymbol{b} - \boldsymbol{A}_N(f)\boldsymbol{x}^{(k)}\|_2 / \|\boldsymbol{b}\|_2$ (solid curve with points),
ii) $\log_{10} \|\boldsymbol{c} - \boldsymbol{B}_N\boldsymbol{y}^{(k)}\|_2 / \|\boldsymbol{c}\|_2$ (solid curve),
iii) $\log_{10}\ |\tilde{\boldsymbol{r}}_k\|_2 / \|\tilde{\boldsymbol{r}}_0\|_2$ (dashed curve)
after a fixed number of iterations $\leq 35$. Here i) is the computational error with respect to

the original problem. Except of PCG, ii) shows the computational error with respect to the preconditioned systems, i.e. by (6.1) – (6.4),

$$\begin{aligned} \boldsymbol{B}_N &:= \boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{A}_N(f) \\ \boldsymbol{c} &:= \boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{b}\,,\ \boldsymbol{y}^{(k)} \ := \ \boldsymbol{x}^{(k)}\,, \end{aligned} \qquad \text{in the case of MINRES}$$

$$\begin{aligned} \boldsymbol{B}_N &:= \boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}}\boldsymbol{A}_N(f)\boldsymbol{M}_{N,f}(|f|)^{-\frac{1}{2}} \\ \boldsymbol{c} &:= \boldsymbol{M}_{N,|f|}(f)^{-\frac{1}{2}}\,\boldsymbol{b}\,,\ \boldsymbol{y}^{(k)} \ := \ \boldsymbol{M}_{N,f}(|f|)^{\frac{1}{2}}\boldsymbol{x}^{(k)}\,, \end{aligned} \qquad \text{in the case of CGNE}$$

$$\begin{aligned} \boldsymbol{B}_N &:= (\boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{A}_{N,f}(f))^*\,(\boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{A}_N(f)) \\ \boldsymbol{c} &:= (\boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{A}_N(f))^*\,\boldsymbol{M}_{N,f}(|f|)^{-1}\,\boldsymbol{b}\,,\ \boldsymbol{y}^{(k)} \ := \ \boldsymbol{x}^{(k)}\,, \end{aligned} \qquad \text{in the case of CG}$$

$$\begin{aligned} \boldsymbol{B}_N &= \boldsymbol{A}_N(f)^*\,\boldsymbol{A}_N(f) \\ \boldsymbol{c} &= \boldsymbol{A}_N(f)^*\,\boldsymbol{b}\,,\ \boldsymbol{y}^{(k)} \ := \ \boldsymbol{x}^{(k)}\,. \end{aligned} \qquad \text{in the case of PCG}$$

Finally, the values in iii) coincide with the values in ii) up to roundoff errors, i.e.

$$\tilde{\boldsymbol{r}}_k \approx \boldsymbol{c} - \boldsymbol{B}_N \boldsymbol{y}^{(k)}\,.$$

The values $\tilde{\boldsymbol{r}}_k$ can be computed during the algorithm without additional effort.
Note that the standard MATLAB 5 implementation of PCG uses i) as stopping criterion.

Indeed, in all algorithms, the dashed line and the solid line are the same during the first iteration steps, but distinguish after a larger number of iterations.
Further, the solid line with points (see i)) doesn't further decrease after certain iterations. We cannot solve $\boldsymbol{A}_{128}(f_1)\boldsymbol{x} = \boldsymbol{b}$ with a relative original residual smaller than $10^{-10}$ by using MINRES or CGNE and smaller than $10^{-6}$ by using PCG and $10^{-5}$ by using CG applied to (6.3). Therefore the latter two methods are not suited for the solution of (1.1) with the considered functions $f$.

Figure 1: Residual norms versus the number of iterations for the matrix $\boldsymbol{A}_{128}(f_1)$ for MINRES applied to (6.1)(left) and for for CGNE applied to (6.2)(right) pst.fig1

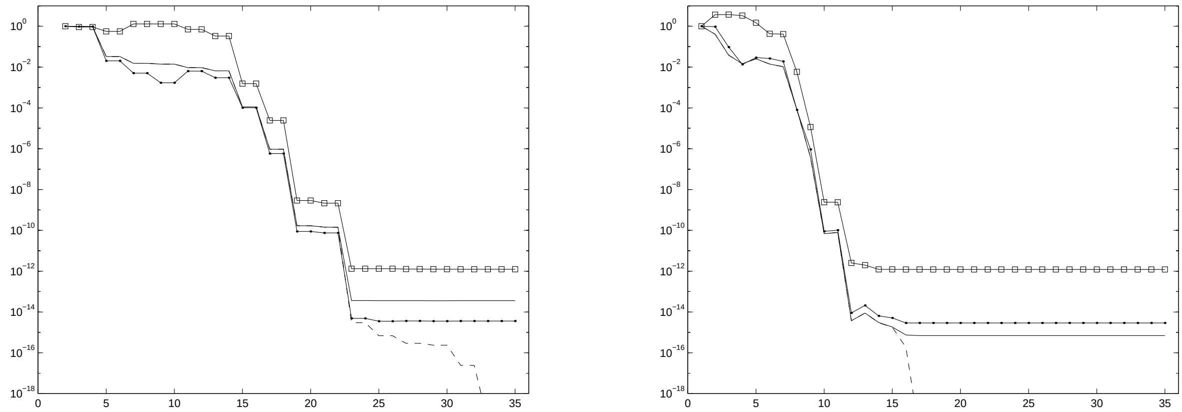
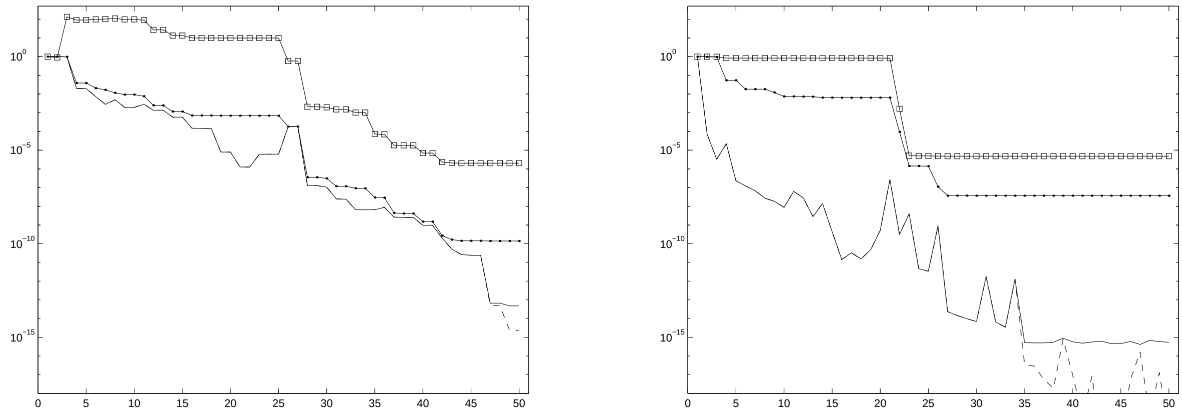



Figure 2: Residual norms versus the number of iterations for the matrix $\boldsymbol{A}_{128}(f_1)$ for PCG-method applied to (6.4)(left) and for CG-method applied to (6.3)(right) pst.fig2

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