Bonnet Pairs in the 3-Sphere

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1. Introduction

Two non-congruent surfaces that are isometric and have the same mean curvature at corresponding points are called a *Bonnet pair of surfaces* or simply a *Bonnet pair*. The interest in such pairs arises from the following considerations: One of the fundamental problems of surface theory is to find invariants which characterize surfaces geometrically. The Bonnet theorem states that a surface is determined up to congruence by its first and second fundamental forms. However, this description has some redundancy since the first and second fundamental forms must satisfy the Gauss and Codazzi equations. So it may be that some pieces of information can be eliminated from this description, but which? It seems wise to retain the metric. But given the metric, we know the Gaussian curvature, i.e. the product of the two principal curvatures. The second fundamental form contains the following extra pieces of information: the mean curvature, i.e. half the sum of the two principal curvatures, and the principal curvature directions. It seems worthwhile to investigate what happens if we do not prescribe the curvature directions and ask the question: Is a surface already determined by its metric and mean curvature?

Bonnet himself [**6**] gave the following answer: In general, a surface is determined up to congruence by its metric and mean curvature. There are only three exceptions:

- *Constant mean curvature surfaces* can be continuously deformed while keeping their metric and mean curvature fixed.
- If a surface is not determined by its metric and mean curvature and the mean curvature is not constant, then there is either a one-parameter familiy of non-congruent isometric surfaces with the same mean curvature (*Bonnet surfaces*), or
- there is exactly one non-congruent isometric surface with the same mean curvature.

Constant mean curvature surfaces form their own field of study, and we will not go in this direction at all.

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Bonnet surfaces were studied by many mathematicians; an incomplete list includes Hazzidakis [**10**], Graustein [**9**], Cartan [**7**], Chern [**8**], Kenmotsu [**13**], Roussos [**16**], Bobenko and Eitner [**4, 5**]. Cartan develops a more or less complete local theory. Bobenko and Eitner describe Bonnet surfaces using Painlevé equations. This makes it possible to give a global classification.

Much less is known about the case in which there are exactly two non-congruent surfaces with the same mean curvature. However, there are trivial examples: helicoidal surfaces [**15**]. Suppose $F(x, y)$ is an immersion in $\mathbf{R}^3$ such that there is a non-compact one parameter subgroup $T(s)$ of the group of rigid motions with $F(x, y + s) = T(s)F(x, y)$. Then $-F(x, -y)$ is by symmetry an isometric immersion with the same mean curvature function. But there is in general no isometry S of $\mathbb{R}^3$, with $-F(x, -y) = SF(x, y)$. Note that if the helicoidal surface degenerates to a surface of rotation, the the Bonnet pair degenerates to a single isothermic surface.

There is a relationship between Bonnet pairs in $\mathbb{R}^3$ and isothermic surfaces in S^3 that was discovered by Servant [**17**]. It is due to an equivalence of the Gauss and Codazzi equations specialized to either class of surfaces. This relationship was studied further by Bianchi [**1**] and Jonas [**11**], who consider Darboux transformations of Bonnet pairs. More recently, this relationship was rediscovered by Kamberov, Pedit and Pinkall [**12**], using their quaternionic calculus.

Lawson and Tribuzy [**14**] show among other things that there are no Bonnet pairs that are topologically spheres. It is an open question whether there are any compact Bonnet pairs. Bobenko [**3**] takes some steps towards attacking this problem by treating Bonnet pairs as integrable systems.

In this article, we study Bonnet pairs not in $\mathbf{R}^3$, but in the three dimensional sphere S^3. The author is not aware of any previous work done on Bonnet pairs in S^3. This theory is at first completely analogous to the $\mathbb{R}^3$-theory. However, it turns out that there are trivial compact examples. We present a duality transformation by which a Bonnet pair in S^3 is transformed into another such pair.

2. Surfaces in the 3-Sphere and Quaternionic Frames

In this section we outline the use of complex notation and quaternionic frames in the analytic description of conformally immersed surfaces in the 3-sphere. See also [**2**]. Throughout this paper, let D be a simply connected domain in $\mathbb{C}$ with coordinate $z = x + iy$.

Suppose $F : D \to S^3 \subset \mathbb{R}^4$ is a conformal immersion with normal $N : D \to S^3$. Conformality is equivalent to

$$\langle F_z, F_z \rangle = 0, \qquad \langle F_z, F_{\bar{z}} \rangle = \frac{1}{2}\, e^u, \qquad \langle F_{\bar{z}}, F_{\bar{z}} \rangle = 0,$$

for some real valued function u. (We use the Wirtinger operators $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ and denote partial differentiation by subscripts. The scalar products are complexified bilinearly: $\langle x, y \rangle = \sum_1^n x_k y_k$ for $x, y \in \mathbb{R}^n$ or $\mathbb{C}^n$.)

The first fundamental form is therefore

$$\langle dF,\, dF\rangle = e^u\, dz\, d\bar{z},$$

and the second fundamental form is

$$-\langle dF,\, dN\rangle = Q\, dz^2 + e^u H\, dz\, d\bar{z} + \overline{Q}\, d\bar{z}^2,$$

where

$$Q = \langle F_{zz}, N\rangle \quad \text{and} \quad H = 2e^{-u}\langle F_{z\bar{z}}, N\rangle.$$

The $(2,0)$-form $Q\, dz^2$ is the Hopf differential and H is the mean curvature function of F. The zeros of Q are the umbilics of F. If (and only if) F is parametrized by curvature-line coordinates, then Q takes either only real or only purely imaginary values. One obtains the following frame equations:

$$\begin{array}{lclcccccc} F_{zz} & = & & & u_z\, F_z & & & + & Q\, N \\ F_{z\bar{z}} & = & -\frac{1}{2}e^u\, F & & & & & + & \frac{1}{2}e^u H\, N \\ F_{\bar{z}\bar{z}} & = & & & & & u_{\bar{z}}\, F_{\bar{z}} & + & \overline{Q}\, N \\ N_z & = & & - & H\, F_z & - & 2e^{-u} Q\, F_{\bar{z}} & & \\ N_{\bar{z}} & = & & - & 2e^{-u}\overline{Q}\, F_z & - & H\, F_{\bar{z}} & & \end{array} \tag{2.1}$$

Now we introduce quaternionic frames. Identify $\mathbb{R}^4$ with the algebra $\mathbb{H}$ of quaternions, so that the canonical basis of $\mathbb{R}^4$ corresponds to the quaternions $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$. Further, use the following representation of the quaternions as complex $2{\times}2$-matrices:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathbf{i} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},\ \mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \mathbf{k} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Then the special unitary group $SU(2)$ is just the unit sphere in $\mathbb{H}$. For $(\Phi_1, \Phi_2) \in SU(2) \times SU(2)$, the linear map $\mathbb{H} \to \mathbb{H}$, $X \mapsto \Phi_2^{-1} X \Phi_1$ is orientation preserving and orthogonal. One obtains thus a right action of $SU(2) \times SU(2)$ on $\mathbb{R}^4$. It defines a two-to-one Lie-group anti-homomorphism onto $SO(4)$, the kernel being $\{(\mathbf{1}, \mathbf{1}), (-\mathbf{1}, -\mathbf{1})\}$.

By a *quaternionic frame* of F we mean a map

$$(\Phi_1, \Phi_2) : D \to SU(2) \times SU(2)$$

with

$$\begin{array}{lcll} F & = & & {\Phi_2}^{-1}\mathbf{1}\Phi_1, \\ F_x & = & e^{u/2} & {\Phi_2}^{-1}\,\mathbf{i}\,\Phi_1, \\ F_y & = & e^{u/2} & {\Phi_2}^{-1}\,\mathbf{j}\,\Phi_1, \\ N & = & & {\Phi_2}^{-1}\mathbf{k}\Phi_1. \end{array}$$

The last equation could have been omitted in this definition, because it follows from the others. Instead of the second and third equation, it is sometimes helpful to use the equivalent

$$\begin{array}{lcl} F_z & = & e^{u/2}\ {\Phi_2}^{-1} \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \Phi_1, \\ F_{\bar{z}} & = & e^{u/2}\ {\Phi_2}^{-1} \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} \Phi_1. \end{array}$$

This setup leads to the fundamental theorem of surface theory in the following guise.

PROPOSITION 1. *The metric $e^u\,dz\,d\bar z$, mean curvature function H, and Hopf differential $Q\,dz^2$ of a conformal immersion into S^3 satisfy the compatibility conditions*

$$\text{(Gauss)} \qquad u_{z\bar z} + \frac{1}{2}e^u(H^2+1) - 2e^{-u}|Q|^2 = 0,$$

$$\text{(Codazzi)} \qquad 2e^{-u/2}Q_{\bar z} = e^{u/2}H_z$$

and determine the immersion up to congruence.

Conversely, if the real valued functions u and H and the complex valued function Q satisfy these compatibility conditions, then the systems

$$\begin{aligned}\Phi_{1z} &= U_1\Phi_1 \\ \Phi_{1\bar z} &= V_1\Phi_1\end{aligned} \qquad \text{and} \qquad \begin{aligned}\Phi_{2z} &= U_2\Phi_2 \\ \Phi_{2\bar z} &= V_2\Phi_2\end{aligned} \tag{2.2}$$

with

$$\begin{aligned} U_1 &= \begin{pmatrix} u_z/4 & -e^{-u/2}Q \\ \frac{1}{2}e^{u/2}(H-i) & -u_z/4 \end{pmatrix}, \; V_1 = \begin{pmatrix} -u_{\bar z}/4 & -\frac{1}{2}e^{u/2}(H+i) \\ e^{-u/2}\overline{Q} & u_{\bar z}/4 \end{pmatrix}, \\ U_2 &= \begin{pmatrix} u_z/4 & -e^{-u/2}Q \\ \frac{1}{2}e^{u/2}(H+i) & -u_z/4 \end{pmatrix}, \; V_2 = \begin{pmatrix} -u_{\bar z}/4 & -\frac{1}{2}e^{u/2}(H-i) \\ e^{-u/2}\overline{Q} & u_{\bar z}/4 \end{pmatrix}, \end{aligned} \tag{2.3}$$

are compatible. Solutions $\Phi_1, \Phi_2 : D \to SU(2)$ yield a conformal immersion $F = {\Phi_2}^{-1}\Phi_1$ into S^3 with metric $e^u\,dz\,d\bar z$, mean curvature H and Hopf differential $Q\,dz^2$.

PROOF-OUTLINE. We omit the somewhat lengthy calculations needed to prove this. The main ingredient of the proof is the Maurer-Cartan lemma stating that a system $\Phi_z = U\Phi$, $\Phi_{\bar z} = V\Phi$ is compatible if and only if $U_{\bar z} - V_z + [U,V] = 0$. First, suppose (Φ_1,Φ_2) is a quaternionic frame of a conformal immersion in S^3 and define U and V by equations (2.2). Use the frame equations (2.1) to derive equations (2.3). Finally, show by straightforward calculation that the Gauss and Codazzi equations are equivalent to $U_{k\,\bar z} - V_{k\,z} + [U_k, V_k] = 0$. This also takes care of the converse. □

3. Bonnet Pairs

A Bonnet pair in S^3 is a pair of immersions that have the same metric and mean curvature, but are not congruent. Hence their Hopf differentials must be different. We specialize proposition 1 to Bonnet pairs.

PROPOSITION 2. *Suppose $F, \widetilde F : D \to S^3$ are two immersions with the same metric $e^u\,dz\,d\bar z$ and mean curvature function H, but non-identical Hopf differentials $Q\,dz^2$ and $\widetilde Q\,dz^2$. Let $Q_1 = i(Q-\widetilde Q)/2$ and $Q_2 = (Q+\widetilde Q)/2$, so that $Q = Q_2 - iQ_1$*

and $\widetilde{Q} = Q_2 + iQ_1$. *Then the following compatibility conditions are satisfied:*

$$u_{z\bar{z}} + \frac{1}{2}e^u(H^2+1) - 2e^{-u}(|Q_1|^2 + |Q_2|^2) = 0, \tag{3.1}$$

$$Q_1\overline{Q}_2 - \overline{Q}_1 Q_2 = 0, \tag{3.2}$$

$$2e^{-u/2}Q_{2\,\bar{z}} - e^{u/2}H_z = 0, \tag{3.3}$$

$$Q_{1\,\bar{z}} = 0. \tag{3.4}$$

Conversely, suppose the functions $u, H : D \to \mathbb{R}$, $Q_1, Q_2 : D \to \mathbb{C}$ *satisfy the above equations. Then the systems*

$$\begin{Bmatrix} \Phi_{1z} = U_1\Phi_1 \\ \Phi_{1\bar{z}} = V_1\Phi_1 \end{Bmatrix}, \begin{Bmatrix} \Phi_{2z} = U_2\Phi_2 \\ \Phi_{2\bar{z}} = V_2\Phi_2 \end{Bmatrix}, \begin{Bmatrix} \widetilde{\Phi}_{1z} = \widetilde{U}_1\widetilde{\Phi}_1 \\ \widetilde{\Phi}_{1\bar{z}} = \widetilde{V}_1\widetilde{\Phi}_1 \end{Bmatrix}, \begin{Bmatrix} \widetilde{\Phi}_{2z} = \widetilde{U}_2\widetilde{\Phi}_2 \\ \widetilde{\Phi}_{2\bar{z}} = \widetilde{V}_2\widetilde{\Phi}_2 \end{Bmatrix},$$

with

(3.5)

$$U_1 = \begin{matrix} u_z/4 & -e^{-u/2}(Q_2 - iQ_1) \\ \frac{1}{2}e^{u/2}(H-i) & -u_z/4 \end{matrix}, \quad V_1 = \begin{matrix} -u_{\bar{z}}/4 & -\frac{1}{2}e^{u/2}(H+i) \\ e^{-u/2}(\overline{Q}_2 + i\overline{Q}_1) & u_{\bar{z}}/4 \end{matrix},$$

$$U_2 = \begin{matrix} u_z/4 & -e^{-u/2}(Q_2 - iQ_1) \\ \frac{1}{2}e^{u/2}(H+i) & -u_z/4 \end{matrix}, \quad V_2 = \begin{matrix} -u_{\bar{z}}/4 & -\frac{1}{2}e^{u/2}(H-i) \\ e^{-u/2}(\overline{Q}_2 + i\overline{Q}_1) & u_{\bar{z}}/4 \end{matrix},$$

$$\widetilde{U}_1 = \begin{matrix} u_z/4 & -e^{-u/2}(Q_2 + iQ_1) \\ \frac{1}{2}e^{u/2}(H-i) & -u_z/4 \end{matrix}, \quad \widetilde{V}_1 = \begin{matrix} -u_{\bar{z}}/4 & -\frac{1}{2}e^{u/2}(H+i) \\ e^{-u/2}(\overline{Q}_2 - i\overline{Q}_1) & u_{\bar{z}}/4 \end{matrix},$$

$$\widetilde{U}_2 = \begin{matrix} u_z/4 & -e^{-u/2}(Q_2 + iQ_1) \\ \frac{1}{2}e^{u/2}(H+i) & -u_z/4 \end{matrix}, \quad \widetilde{V}_2 = \begin{matrix} -u_{\bar{z}}/4 & -\frac{1}{2}e^{u/2}(H-i) \\ e^{-u/2}(\overline{Q}_2 - i\overline{Q}_1) & u_{\bar{z}}/4 \end{matrix}$$

are all compatible. Solutions $\Phi_1, \Phi_2, \widetilde{\Phi}_1, \widetilde{\Phi}_2, : D \to SU(2)$ *yield conformal immersions* $F = \Phi_2{}^{-1}\Phi_1$, $\widetilde{F} = \widetilde{\Phi}_2^{-1}\widetilde{\Phi}_1$ *with metric* $e^u\,dz\,d\bar{z}$, *mean curvature* H, *and Hopf differentials* $(Q_2 - iQ_1)\,dz^2$ *and* $(Q_2 + iQ_1)\,dz^2$. *Hence, if* Q_1 *does not vanish identically, they form a Bonnet pair.*

Proof. Subtract the Gauss equations of F and $\widetilde{F}$ to obtain $|Q| = |\widetilde{Q}|$ and hence equations (3.1) and (3.2). Adding and subtracting the Codazzi equations gives (3.3) and (3.4). The rest of the proposition follows immediately from proposition 1. □

The following proposition is obtained in exactly the same way as in the case of Bonnet pairs in $\mathbb{R}^3$; see [**3**].

Proposition. *The immersions of a Bonnet pair in* S^3 *have umbilics at corresponding points. They are the zeros of a not identically vanishing holomorphic quadratic differential and therefore isolated.*

There are no Bonnet pairs that are spheres. If Bonnet pair tori exist, they have no umbilics. If Bonnet pairs of genus $g > 1$ *exist, they have* $4g - 4$ *umbilics, counting multiplicities.*

Standard arguments also imply:

Lemma. *Let* $F, \widetilde{F} : D \to S^3$ *form a Bonnet pair with fundamental invariants as in proposition 2, and assume that they have no umbilics. Then the conformal parametrization can be chosen such that* $Q_1 \equiv \frac{1}{2}$ *and* Q_2 *is real valued.*

The development so far has been completely parallel to the case of Bonnet pairs in $\mathbb{R}^3$. With the following theorem, however, we depart from the common features and turn to the properties peculiar to Bonnet pairs in S^3. As in the $\mathbb{R}^3$ case, helicoidal surfaces provide trivial examples of Bonnet pairs. But in S^3, such surfaces can be compact. That is:

THEOREM. *There are compact Bonnet pairs in S^3 which are helicoidal immersed tori.*

Next we introduce a duality transformation for Bonnet pairs.

THEOREM. *Suppose $F, \widetilde{F} : D \to S^3$ form a Bonnet pair with fundamental invariants as in proposition 2. Assume the immersions have no umbilics and are parametrized such that $Q = \frac{1}{2}(q - i)\,dz^2$ and $\widetilde{Q} = \frac{1}{2}(q + i)\,dz^2$ with real-valued q. Let (Φ_1, Φ_2) and $(\widetilde{\Phi}_1, \widetilde{\Phi}_2)$ be quaternionic frames for F and $\widetilde{F}$.*

Then $G = \widetilde{\Phi}_1^{-1}\Phi_1$ and $\widetilde{G} = \widetilde{\Phi}_2^{-1}\Phi_2$ are conformal immersions into S^3 forming another—the dual*—Bonnet pair with metric $e^{-u}\,dz\,d\bar{z}$, mean curvature q, and Hopf differentials $\frac{1}{2}(H - i)\,dz^2$ and $\frac{1}{2}(H + i)\,dz^2$. The dual Bonnet pair to $G, \widetilde{G}$ is again $F, \widetilde{F}$.*

PROOF. The idea is to show that $(\mathbf{j}\Phi_1, \mathbf{j}\widetilde{\Phi}_1)$ is a quaternionic frame of an immersion G with the required fundamental invariants. Similarly, $(\mathbf{j}\Phi_2, \mathbf{j}\widetilde{\Phi}_2)$ is a quaternionic frame for $\widetilde{G}$. To this end, substitute $\frac{1}{2}$ for Q_1 and q for Q_2 in equations (3.5) and compare these matrices to the corresponding 'U' and 'V' matrices of $\mathbf{j}\Phi_1$, $\mathbf{j}\widetilde{\Phi}_1$, $\mathbf{j}\Phi_2$ and $\mathbf{j}\widetilde{\Phi}_2$. For example, for Φ_1, calculate $(\mathbf{j}\Phi_1)_z(\mathbf{j}\Phi_1)^{-1} = -\mathbf{j}U_1\mathbf{j}$ and $(\mathbf{j}\Phi_1)_{\bar{z}}(\mathbf{j}\Phi_1)^{-1} = -\mathbf{j}V_1\mathbf{j}$ and compare them to U_1 and V_1. □

COROLLARY. *The dual Bonnet pair satisfies the following equations.*

$$F = \widetilde{G}^{-1}\widetilde{F}\,G,$$
$$F_x = \widetilde{G}^{-1}\widetilde{F}_x\,G,$$
$$F_y = \widetilde{G}^{-1}\widetilde{F}_y\,G.$$

In other words, $(G, \widetilde{G})$, interpreted as a map into $SU(2) \times SU(2)$, rotates the frame of $\widetilde{F}$ into the frame of F.

The following interesting questions are left unanswered: Can the theory of Bonnet pairs in S^3 be generalized to encompass Bonnet pairs in $\mathbb{R}^3$? Is there an integrable system connected to Bonnet pairs in S^3? Are there compact examples apart from helicoidal tori?

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