Homothetic triangle contact representations

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Abstract. We prove that every 4-connected planar triangulation admits a contact representation by homothetic triangles.
There is a known proof of this result that is based on the Convex Packing Theorem by Schramm, a general result about contact representations of planar triangulations by convex shapes. But our approach makes use of the combinatorial structure of triangle contact representations in terms of Schnyder woods. We start with an arbitrary Schnyder wood and produce a sequence of Schnyder woods via face flips. We show that at some point the sequence has to reach a Schnyder wood describing a representation by homothetic triangles.

Keywords: Contact representation · Schnyder wood · Triangle · Planar triangulation

1 Introduction

A *triangle contact system* $\mathcal{T}$ is a finite system of triangles in the plane such that any two triangles intersect in at most one point. The contact system is *nondegenerate* if every contact involves exactly one corner of a triangle. The graph $G^*(\mathcal{T})$ is the plane graph that has a vertex for every triangle of $\mathcal{T}$ and an edge for every contact of two triangles in $\mathcal{T}$. For a given plane graph G and a triangle contact system $\mathcal{T}$ with $G^*(\mathcal{T}) = G$ we say that $\mathcal{T}$ is a *triangle contact representation* of G.

The main goal of this paper will be to prove the following result.

Theorem 1 ([5]). *Let G be a 4-connected planar triangulation. Then there is a triangle contact representation of G by homothetic triangles.*

The original proof of Theorem 1 in [5][1] makes use of the following theorem by Schramm.

Theorem 2 (Convex Packing Theorem [10]). *Let G be a triangulation with outer face $\{a, b, c\}$. Further let C be a simple closed curve in the plane partitioned into arcs $\mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c$ and for each interior vertex v of G let $\mathcal{Q}_v$ be a convex set in the plane containing more than one point. Then there is a contact representation of a supergraph of G (on the same vertex set, but possibly with more edges) where each interior vertex v is represented by a homothetic copy of its prototype $\mathcal{Q}_v$ and each outer vertex w by the arc P_w.*

[1] The journal version [6] does not contain this proof.

If we want to calculate a homothetic triangle contact representation of G efficiently, this theorem does not help since it is purely existential. On the other hand, Felsner [2] introcuced a combinatorial heuristic that calculates triangle contact representations quite fast in practical experiments [8]. However, to the best of our knowledge, it is not known whether this heuristic terminates for every instance, nor whether it has a good (e.g., polynomial) running time if it terminates.

The heuristic starts by guessing the combinatorial structure of the contact representation in the form of a Schnyder wood. Then a system of linear equations is solved whose variables correspond to the lengths of the segments of the triangles in the contact representation. If the solution is nonnegative, this yields the intended contact representation. Otherwise, the negative variables of the solution can be used as sign-posts indicating how to change the Schnyder wood for another try.

Our new proof of Theorem 1 is based on the theoretical background of this heuristic. Therefore it might help to better understand this heuristic in the future. Felsner and Francis [3] even explicitly ask for a proof of Theorem 1 by this approach. Further, our proof motivates a new heuristic for calculating homothetic triangle contact representations.

A substantial part of this work originates in the author's Masters thesis [12]. In this thesis with a similar approach also the existence of contact representations of 5-connected planar triangulations by homothetic squares has been proved. But in that case there are other known proofs which are not based on the Convex Packing Theorem [7, 11]. That is why we will focus on contact representations by homothetic triangles in this paper.

Let us get back to triangle contact representations. In the case that $G^*(\mathcal{T})$ is a planar triangulation, in $\mathcal{T}$ the inner (i.e., bounded) faces of $G^*(\mathcal{T})$ are also represented by triangles. We denote these by *dual triangles* and for clear distinction the triangles of $\mathcal{T}$ by *primal triangles*.

In Theorem 1 we do not specify what is the shape of the homothetic triangles. The reason is that if we are given a contact representation by homothetic triangles, we can change the shapes of these triangles to homothetic copies of an arbitrary given triangle by a linear transformation of the plane. So we choose to prove the existence of a contact representation by right, isosceles triangles with a horizontal edge at the bottom and a vertical edge at the right hand side. We will even consider a larger class of triangle contact representations. A *right triangle contact representation* is a triangle contact representation by right triangles with a horizontal edge at the bottom and a vertical edge at the right hand side. The *aspect ratio* of such a triangle is the quotient of the lengths of its vertical and its horizontal edge. The *aspect ratio vector* of a right triangle contact representation is the vector of the aspect ratios of its triangles (we assume the vertices of G have a fixed numbering $1, \ldots, n+3$). See Fig. 1 for an example of a right triangle contact representation. Now we can formulate a stronger theorem that implies Theorem 1.

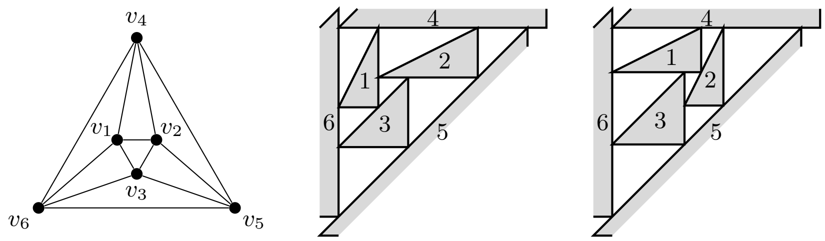

Fig. 1: Two right triangle contact representations of the same graph with aspect ratio vectors $(2, 1/2, 1, 1, 1, 1)$ and $(1/2, 2, 1, 1, 1, 1)$.

Theorem 3. *Let G be a 4-connected triangulation and $\tilde{r} \in \mathbb{R}^{n+3}_{>0}$. Then there is a right triangle contact representation of G with aspect ratio vector $\tilde{r}$.*

The paper is organized as follows: In Section 2 we give an introduction to *Schnyder woods* as the combinatorial structure describing triangle contact representations and recall some known results about them. In Section 3 we describe a variant of the system of linear equations from the heuristic by Felsner for calculating a right triangle contact representation with given Schnyder wood and given aspect ratio vector. As our main contribution, we prepare the prove of Theorem 3 in Section 4 and give the proof in Section 5. In Section 6 we propose a new heuristic for calculating right triangle contact representation based on this proof.

2 Schnyder woods

Schnyder woods are a combinatorial structure on triangulations that play a central role in this paper. They were first introduced by Schnyder [9] under the name of *realizers*.

Definition 1. *Let G be a triangulation with outer vertices $v_{n+1}, v_{n+2}, v_{n+3}$ in clockwise order. Then a* Schnyder wood *of G is an orientation and coloring of the interior edges of G with the colors red, green and blue such that*

- *each edge incident to v_{n+1} is red and incoming, each edge incident to v_{n+2} is green and incoming, and each edge incident to v_{n+3} is blue and incoming,*
- *each inner vertex has in clockwise order exactly one red, green and blue outgoing edge, and in the interval between two outgoing edges there are only incoming edges in the third color (see Fig. 2).*

If we forget about the colors of a Schnyder wood, we obtain a 3-orientation, i.e., each inner vertex has outdegree 3 and the outer vertices have outdegree 0. The converse also holds:

Proposition 1 (de Fraysseix and Ossona de Mendez [13]). *If the graph G is a 3-orientation of a triangulation with outer vertices labeled $v_{n+1}, v_{n+2}, v_{n+3}$*

in clockwise order, then there is a unique way of coloring the interior edges of G to receive a Schnyder wood.

Let $\mathcal{T}$ be a triangle contact system such that $G := G^*(\mathcal{T})$ is a triangulation. If $\mathcal{T}$ is nondegenerate, we can orient each inner edge of G from the triangle whose corner is involved in the contact to the other triangle and obtain an orientation where the outdegree of each inner vertex is at most 3. In the case that $\mathcal{T}$ is degenerate, we can interpret a point where several triangle corners meet as a cyclic sequence of nondegenerate contacts with infinitesimal edge lengths and proceed as in the nondegenerate case. As a consequence of Euler's formula, G has exactly $3n$ inner edges, and therefore the outdegree of each inner vertex has to be exactly 3. Thus $\mathcal{T}$ induces a 3-orientation and hence a Schnyder wood of G. We will call this Schnyder wood the (induced) Schnyder wood of $\mathcal{T}$. Note that the induced Schnyder wood is not unique if $\mathcal{T}$ is degenerate. The following proposition shows that every Schnyder wood is an induced Schnyder wood.

Proposition 2 (de Fraysseix, Ossona de Mendez and Rosenstiehl [14]). *Let G be a triangulation and S a Schnyder wood of G. Then there exists a nondegenerate right triangle contact representation of G with induced Schnyder wood S.*

For right triangle contact representations we can obtain the colors of the edges of the associated Schnyder wood also directly, without using Proposition 1. We color an edge red if it corresponds to the upper corner of a triangle, green if it corresponds to the right lower corner of a triangle, and blue if it corresponds to the left lower corner of a triangle. See Fig. 3 for an example.

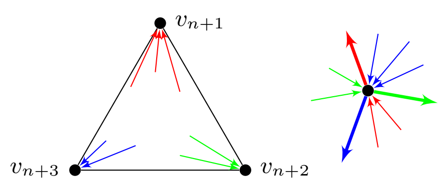

Fig. 2: The local conditions of a Schnyder wood.

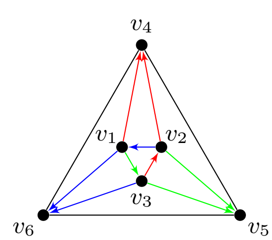

Fig. 3: The Schnyder wood induced by the first example of Fig. 1.

3 The System of Linear Equations

For the whole section let G be a triangulation with inner vertices $v_1, \ldots, v_n$ and outer vertices $v_{n+1}, v_{n+2}, v_{n+3}$ in clockwise order, let S be a Schnyder wood of G

and let $r \in \mathbb{R}^{n+3}_{>0}$ be an aspect ratio vector for G. We will now describe a system of linear equations for calculating the edge lengths of a right triangle contact representation of G with aspect ratio vector r and induced Schnyder wood S. This system has been introduced by Felsner [2] and studied by Rucker [8] for the special case $r = (1, \ldots, 1)$. All results in this section are due to them.

For each inner vertex v of G we have a variable x_v which represents the width of the corresponding primal triangle, and for each inner face f of G a variable x_f which represents the width of the corresponding dual triangle. For a vertex v of G we denote by $\delta_r(v)$ the set of incident faces of v which are located in the interval between the green and blue outgoing edge of v. Analogously we define the sets $\delta_g(v)$ and $\delta_b(v)$. In a right triangle contact representation a primal triangle T hands down his aspect ratio to each dual triangle whose hypotenuse is contained in the hypotenuse of T. If T corresponds to the vertex v of G, these are exactly the dual triangles corresponding to the faces in $\delta_g(v)$. Therefore, if we are given the Schnyder wood and the aspect ratio vector of a right triangle contact representation, we are implicitly also given the aspect ratios of the dual triangles. We denote the aspect ratio of a dual triangle corresponding to the face f of G by r_f. Now we can write down the equation system:

$$\sum_{f \in \delta_r(v_{n+1})} x_f = 1 \ , \tag{1}$$

$$\sum_{f \in \delta_r(v_i)} x_f - x_{v_i} = 0 \ , \quad i = 1, \ldots, n \ , \tag{2}$$

$$\sum_{f \in \delta_g(v_i)} x_f - x_{v_i} = 0 \ , \quad i = 1, \ldots, n \ , \tag{3}$$

$$\sum_{f \in \delta_b(v_i)} r_f x_f - r_{v_i} x_{v_i} = 0 \ , \quad i = 1, \ldots, n \ . \tag{4}$$

Equation (2) says that the length of the horizontal edge of a primal triangle is equal to the sum of the lengths of the adjacent dual triangles. Equations (3) and (4) say the same for the other two edges of a primal triangle. Note that the sum of the lengths of the diagonal edges of the dual triangles corresponding to the faces in $\delta_g(v_i)$ is equal to the length of the diagonal edge of the primal triangle of v_i if and only if the sum of the lengths of their horizontal edges is equal to the length of the horizontal edge of the primal triangle of v_i. The purpose of (1) is to pick one single solution out of the space of solutions of the apart from that homogeneous equation system. We will also use the shorter notation $A_S(r)x = \mathbf{e_1}$ for the equation system.

Proposition 3. *The system $A_S(r)x = \mathbf{e_1}$ is uniquely solvable.*

Because of the way we chose the equations of the system, it is clear that the existence of a right triangle contact representation of G with Schnyder wood S and aspect ratio vector r implies a nonnegative solution. The following proposition shows that also the converse holds.

Proposition 4. *Let $A_S(r)x = \mathbf{e_1}$. There is a right triangle contact representation of G with induced Schnyder wood S and aspect ratio vector r if and only if $x \geq 0$. If $x \geq 0$, the representation is unique inside the three outer triangles.*

Next we will prove a result about the structure of nonnegative solutions with zero entries.

Definition 2. *Let $A_S(r)x = \mathbf{e_1}$ and $x \geq 0$. Then an edge e of G is called a* transition edge *if it is incident to inner faces f_1 and f_2 of G with $x_{f_1} > 0$ and $x_{f_2} = 0$.*

Lemma 1. *Let $A_S(r)x = \mathbf{e_1}$ and $x \geq 0$. Then the transition edges of G form an edge disjoint union of cycles of length 3. Moreover there is no edge going from a vertex on such a cycle into the interior of this cycle.*

For the sake of completeness we will now briefly describe the heuristic by Felsner [2] for calculating triangle contact representations that is based on the system of linear equations. Felsner only considers homothetic triangle contact representations, but the heuristic can easily be translated to the case of right triangle contact representations with given aspect ratio vector $\tilde{r}$. For the heuristic we need a result similar to Lemma 1 for the case $x \ngeq 0$.

Definition 3. *Let $A_S(r)x = \mathbf{e_1}$. Then an edge e of G is called a* sign-separating edge *if it is incident to inner faces f_1 and f_2 of G with $x_{f_1} \geq 0$ and $x_{f_2} < 0$.*

Lemma 2. *Let $A_S(r)x = \mathbf{e_1}$. Then the sign-separating edges of G form an edge-disjoint union of directed simple cycles.*

The heuristic starts with an arbitrary Schnyder wood S and solves the system of linear equations $A_S(\tilde{r})x = \mathbf{e_1}$. If the solution is nonnegative, we are done and can compute the contact representation due to Proposition 4. Otherwise, we change in S the orientation of all sign-separating edges. Since the set of sign-separating edges is a disjoint union of simple cycles, we obtain a new 3-orientation of G and thus, due to Proposition 1, a new Schnyder wood S'. Now we proceed with solving the system of linear equations for the new Schnyder wood S' and so on.

In experiments by Rucker [8] this heuristic delivered good results (for the case $r = (1, \ldots, 1)$), i.e., it always terminated after a small number of iterations. But in theory we neither know whether it terminates for every instance, nor know any nontrivial bounds for the number of iterations it takes in the case of termination. Also variants of this heuristic have been studied where the Schnyder wood is changed in some other way, but without success.

4 Preparation of the proof of Theorem 3

In this section we will introduce some notation and present some lemmas we need for the proof of Theorem 3. Let G be a 4-connected plane triangulation for the whole section.

4.1 Feasible aspect ratio vectors for a fixed Schnyder wood

Remember that a triangle contact representation is degenerate if corners of several primal triangles meet in a single point.

Definition 4. *Let S be a Schnyder wood. Then $\mathcal{R}_S$ is defined as the set of aspect ratio vectors of nondegenerate right triangle contact representations of G with induced Schnyder wood S, and $\bar{\mathcal{R}}_S$ as the set of aspect ratio vectors of all (possibly degenerate) right triangle contact representations of G with induced Schnyder wood S.*

In Proposition 4 we have seen that $r \in \mathcal{R}_S$ if and only if $x > 0$ where x is the solution of $A_S(r)x = \mathbf{e_1}$. With Cramer's rule and by bounding the degrees of the occurring polynomials (the determinants) we then get the following:

Lemma 3. *There are polynomials $p_1, \dots, p_{3n+1}$ in the variables $r_1, \dots, r_{n+3}$ with $\deg p_j \leq 3n+1$ for each j such that*

$$\mathcal{R}_S = \{r \in \mathbb{R}^{n+3}_{>0} : p_j(r) > 0 \text{ for } j = 1, \dots, 3n+1\} \ ,$$
$$\bar{\mathcal{R}}_S = \{r \in \mathbb{R}^{n+3}_{>0} : p_j(r) \geq 0 \text{ for } j = 1, \dots, 3n+1\} \ .$$

In particular $\mathcal{R}_S$ is an open set and $\bar{\mathcal{R}}_S$ its closure.

The following lemma follows from Lemma 3 and shows that the intersection of $\mathcal{R}_S$ with a line segment decomposes into a bounded number of intervals.

Lemma 4. *Let $r_0, r_1 \in \mathbb{R}^{n+3}_{>0}$ be two distinct aspect ratio vectors and for each $0 \leq t \leq 1$ let $r_t := (1-t)r_0 + tr_1$. Then there are open intervals $I_1, \dots, I_k$ with $k \leq (3n+1)\lfloor \frac{3n+1}{2} \rfloor + 1$ such that*

$$I_1 \cup \dots \cup I_k = \{t \in \mathbb{R} : 0 < t < 1, r_t \in \mathcal{R}_S\} \ ,$$
$$\bar{I}_1 \cup \dots \cup \bar{I}_k \subseteq \{t \in \mathbb{R} : 0 \leq t \leq 1, r_t \in \bar{\mathcal{R}}_S\} \ .$$

4.2 Neighboring Schnyder woods

Let G be a planar triangulation. If S and S' are two Schnyder woods of G, then S' can be obtained from S by changing the orientation of the edges in some edge-disjoint directed simple cycles of S. We introduce some notation for Schnyder woods whose difference is small in this sense.

Definition 5. *We call two Schnyder woods S and S'* neighboring *if the corresponding 3-orientations differ in a single facial cycle C. In this case we call C the* difference cycle *of S and S'.*

The set of Schnyder woods of a fixed graph has been thoroughly studied and it is well known that it has the structure of a distributive lattice with the cover relation being exactly this neighboring relation [1].

Proposition 5. *Let S and S' be two neighboring Schnyder woods and let f be the face of G bounded by the difference cycle of S and S'. Further let r be an aspect ratio vector, and let x and x' be the solutions of $A_S(r)x = \mathbf{e_1}$ and $A_{S'}(r)x' = \mathbf{e_1}$. Then the variables x_f and x'_f corresponding to the face f have different signs, or $x_f = x'_f = 0$.*

Corollary 1. *Let S and S' be two neighboring Schnyder woods. Furthermore let $r \in \mathcal{R}_S$. Then $r \notin \bar{\mathcal{R}}_{S'}$.*

This can be seen as a weak variant of the following conjecture which is motivated by the fact that contact representations of 5-connected triangulations by homothetic squares are unique [7, 11].

Conjecture 1. Let G be a 4-connected triangulation and $\tilde{r} \in \mathbb{R}^{n+3}_{>0}$. Then the right triangle contact representation of G with aspect ratio vector $\tilde{r}$ is unique inside the three outer triangles up to scaling.

The strategy of the proof of Theorem 3 will be to move along a line segment of aspect ratio vectors keeping the invariant that there exits a right triangle contact representation with the current aspect ratio vector. In this process, the Schnyder wood of this triangle contact representation will stay the same for a whole subsegment of this line segment. The following lemma allows us to switch to a neighboring Schnyder wood if the current one does not work any more.

Lemma 5. *Let $\{s_t = (1-t)s_0 + ts_1 : 0 \leq t \leq 1\}$ be a line segment of aspect ratio vectors. Let $0 < t_0 < 1$ such that $s_{t_0} \in \bar{\mathcal{R}}_S \setminus \mathcal{R}_S$, in the corresponding solution of the equation system only one face variable x_f is zero, and there is an $\varepsilon > 0$ such that for each $t_0 < t \leq t_0 + \varepsilon$ we have $s_t \notin \bar{\mathcal{R}}_S$. Let S' be the neighboring Schnyder wood of S whose difference cycle is the facial cycle of the face f. Then there is an $\varepsilon' > 0$ such that $s_{t_0+\varepsilon'} \in \mathcal{R}_{S'}$.*

Proof. The matrices $A_S(s_{t_0})$ and $A_{S'}(s_{t_0})$ only differ in the column corresponding to the variable x_f. In the solution of $A_S(s_{t_0})x = \mathbf{e_1}$ we have $x_f = 0$. Therefore the solution of $A_S(s_{t_0})x = \mathbf{e_1}$ is also a solution of $A_{S'}(s_{t_0})x = \mathbf{e_1}$, or in other words the solutions of $A_S(s_{t_0})x = \mathbf{e_1}$ and $A_{S'}(s_{t_0})x = \mathbf{e_1}$ are equal.

Now let us view the solutions of $A_{S'}(s_t)x = \mathbf{e_1}$ for t slightly larger than t_0. Since for the aspect ratio vector s_{t_0} all variables except x_f are strictly positive, there is, due to continuity, an $\varepsilon'' > 0$ such that for all aspect ratio vectors s_t with $t_0 \leq t \leq t_0 + \varepsilon''$ all variables except x_f are strictly positive. Due to Proposition 5 we have $x_f > 0$ for all aspect ratio vectors s_t with $t_0 < t \leq t_0 + \min\{\varepsilon, \varepsilon''\}$. Therefore the choice $\varepsilon' := \min\{\varepsilon, \varepsilon''\}$ fulfills $s_{t_0+\varepsilon'} \in \mathcal{R}_{S'}$. □

Since Lemma 5 can only be applied if we run into a degenerate contact representation with only one single degenerate face, we need a more general lemma for the proof of Theorem 3. By $B(m, \rho)$ and $B^\circ(m, \rho)$ we denote the closed and open ball with center m and radius ρ.

Lemma 6. *Let $\{s_t : 0 \leq t \leq 1\}$ be a line segment of aspect ratio vectors. If there is a $0 < t_0 < 1$ such that $s_{t_0} \in \bar{\mathcal{R}}_S \setminus \mathcal{R}_S$ and an $\varepsilon > 0$ such that for each $t_0 - \varepsilon \leq t < t_0$ we have $s_t \in \mathcal{R}_S$, then for each $\varepsilon' > 0$ there is an aspect ratio vector $r \in B(s_{t_0}, \varepsilon')$ and a neighboring Schnyder wood S' of S with $r \in \mathcal{R}_{S'}$.*

5 Proof of Theorem 3

Theorem 3. *Let G be a 4-connected triangulation and $\tilde{r} \in \mathbb{R}^{n+3}_{>0}$. Then there is a right triangle contact representation of G with aspect ratio vector $\tilde{r}$.*

Proof. We assume there is no right triangle contact representation of G with aspect ratio vector $\tilde{r}$. The idea of the proof is to construct under this assumption a line segment contradicting Lemma 4. For that we will construct an infinite sequence $(S_i)_{i\geq 0}$ of Schnyder woods, two sequences $(r_i)_{i\geq 0}$ and $(r'_i)_{i\geq 0}$ of aspect ratio vectors and two sequences $(\varepsilon_i)_{i\geq 0}$ and $(\varepsilon'_i)_{i\geq 0}$ of positive real numbers fulfilling the following invariants:

(I1) For each $r \in B(r_i, \varepsilon_i)$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector r and Schnyder wood S_i.
(I2) For each $r' \in B(r'_i, \varepsilon'_i)$ the line segment $\{(1-t)r' + t\tilde{r} : 0 \leq t \leq 1\}$ intersects the balls $B(r_0, \varepsilon_0), \ldots, B(r_i, \varepsilon_i)$ in this order (with increasing t).
(I3) The points r'_i, r_i and $\tilde{r}$ are collinear and aligned in this order.
(I4) The Schnyder woods S_i and S_{i+1} are neighboring.

It now remains to show how to construct these sequences and why the existence of these sequences contradicts Lemma 4.

Construction of the sequences Let S_0 be an arbitrary Schnyder wood of G, let $\mathcal{T}_0$ be an arbitrary nondegenerate right triangle contact representation of G with Schnyder wood S_0 (such a contact representation exists due to Proposition 2), and let r_0 be the aspect ratio vector of $\mathcal{T}_0$. Then we know from Lemma 3 that there is a $0 < \varepsilon_0 < 1$ such that for each $r \in B(r_0, \varepsilon_0)$ a nondegenerate right triangle contact representation of G with aspect ratio vector r and Schnyder wood S_0 exists. Furthermore we set $r'_0 := r_0$ and $\varepsilon'_0 := \varepsilon_0$. These initial values obviously fulfill each of the four invariants.

Now we describe how to construct the $(j+1)$th sequence members from the jth ones. We set $s_t := (1-t)r'_j + t\tilde{r}$ for $0 \leq t \leq 1$. Then because of (I3) there is a $0 \leq \hat{t} < 1$ with $s_{\hat{t}} = r_j$. Thus Lemma 4 gives us a $\delta > 0$ such that for each $\hat{t} \leq t < \hat{t} + \delta$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector s_t and Schnyder wood S_j, and a degenerate one with aspect ratio vector $s_{\hat{t}+\delta}$ and Schnyder wood S_j. Because of our assumption we have $\hat{t}+\delta < 1$ and because of $s_{\hat{t}+\delta} \notin B(r_j, \varepsilon_j)$ we have $\|r_j - s_{\hat{t}+\delta}\| > \varepsilon_j$. Now we set

$$\delta' := \min\left\{\left(1 - (\hat{t}+\delta)\right)\varepsilon'_j, \|r_j - s_{\hat{t}+\delta}\| - \varepsilon_j\right\} > 0 \ .$$

Then Lemma 6 gives us an $r_{j+1} \in B^\circ(s_{\hat{t}+\delta}, \delta')$ such that there is a nondegenerate right triangle contact representation of G with aspect ratio vector r_{j+1} and a neighboring Schnyder wood S_{j+1} of S_j. Now we set $r'_{j+1} := r'_j + \frac{1}{1-(\hat{t}+\delta)}(r_{j+1} - s_{\hat{t}+\delta})$. Then

$$\|r'_{j+1} - r'_j\| = \frac{1}{1-(\hat{t}+\delta)}\|r_{j+1} - s_{\hat{t}+\delta}\| < \frac{1}{1-(\hat{t}+\delta)}\delta' \leq \varepsilon'_j$$

and therefore we have $r'_{j+1} \in B^\circ(r'_j, \varepsilon'_j)$. Moreover Lemma 3 gives us an $0 < \varepsilon_{j+1} < \delta' - \|r_{j+1} - s_{\hat{t}+\delta}\|$ such that for each $r \in B(r_{j+1}, \varepsilon_{j+1})$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector r and Schnyder wood S_{j+1}. Finally we set $\varepsilon'_{j+1} := \frac{1}{1-(\hat{t}+\delta)}\varepsilon_{j+1}$. See Fig. 4 for an illustration of the construction.

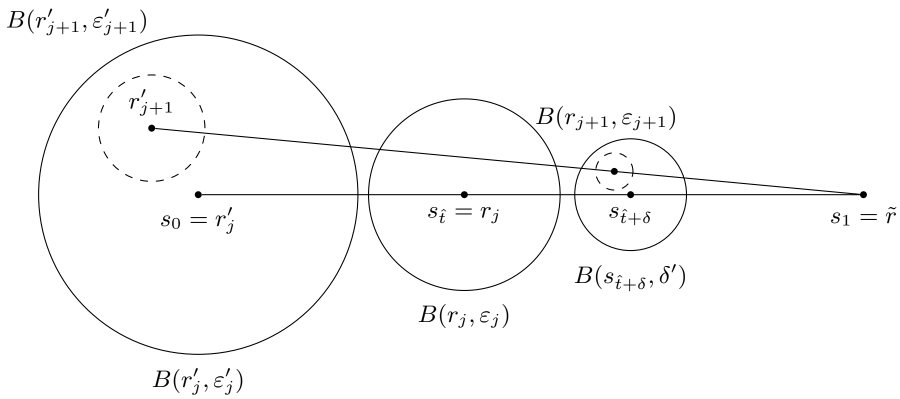

Fig. 4: The construction of the new sequence members.

Clearly the invariants (I1), (I3) and (I4) are fulfilled again. Because of $B(r'_{j+1}, \varepsilon'_{j+1}) \subseteq B(r'_j, \varepsilon'_j)$ for each $r' \in B(r'_{j+1}, \varepsilon'_{j+1})$ the line segment $\{(1-t)r' + t\tilde{r} : 0 \le t \le 1\}$ intersects the balls $B(r_0, \varepsilon_0), \ldots, B(r_j, \varepsilon_j)$ in the right order. From the construction it immediately follows that the ball $B(r_{j+1}, \varepsilon_{j+1})$ is intersected by this line segment, too. Moreover, because of $\delta' \le \|r_j - s_{\hat{t}+\delta}\| - \varepsilon_j$ and $B(r_{j+1}, \varepsilon_{j+1}) \subseteq B^\circ(s_{\hat{t}+\delta}, \delta')$ the intersection point with $B(r_{j+1}, \varepsilon_{j+1})$ is closer to $\tilde{r}$ than the intersection point with $B(r_j, \varepsilon_j)$. Therefore also (I2) is fulfilled again.

Producing a contradiction Let L be the number of Schnyder woods of G and $c := (3n+1)\lfloor \frac{3n+1}{2} \rfloor + 1$ (this is the bound from Lemma 4). We set $K := Lc + 1$. Then it follows from the pigeonhole principle that there is a Schnyder wood S such that there are indices $0 \le i_1 < \cdots < i_{c+1} \le K$ with $S = S_{i_1} = \cdots = S_{i_{c+1}}$. For $l = 1, \ldots, c+1$ let $\hat{r}_l$ be an intersection point of the line segment $\{(1-t)r'_K + t\tilde{r} : 0 \le t \le 1\}$ and the ball $B(r_{i_l}, \varepsilon_{i_l})$. Thus for $l = 1, \ldots, c+1$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector $\hat{r}_l$ and Schnyder wood S. From Lemma 4 it follows that the intersection of $\{(1-t)r'_K + t\tilde{r} : 0 \le t \le 1\}$ and $\mathcal{R}_S$ is a disjoint union of at most c open intervals. Therefore there is an l such that $\hat{r}_l$ and $\hat{r}_{l+1}$ belong to the same interval. Particularly for each $0 \le \tau \le 1$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector $(1-\tau)\hat{r}_l + \tau\hat{r}_{l+1}$ and Schnyder wood S. Because of (I4) the Schnyder woods $S' := S_{i_l+1}$ and S are neighboring. Moreover, because of (I2) there is a $0 \le \tau' \le 1$ with $(1-\tau')\hat{r}_l + \tau'\hat{r}_{l+1} \in B(r_{i_l+1}, \varepsilon_{i_l+1})$. But then because of (I1) there is also a nondegenerate right triangle contact representation of G with aspect ratio vector $(1-\tau')\hat{r}_l + \tau'\hat{r}_{l+1}$ and Schnyder wood S', contradicting Corollary 1. □

6 A new heuristic

In this section we will present an new heuristic for computing a right triangle contact representation of a given planar triangulation G with a given aspect ratio vector $\tilde{r}$ that is based on our proof of Theorem 3. The idea of the heuristic is to make progress on a line segment $\{r_t = (1-t)r_0 + t\tilde{r} : 0 \le t \le 1\}$ of aspect ratio vectors. By that we mean that in each iteration the largest t increases for that we know a Schnyder wood S with $s_t \in \mathcal{R}_S$.

We introduce some notation. For a Schnyder wood S and an aspect ratio vector r we denote by $x(S,r)$ the solution of $A_S(r)x = \mathbf{e_1}$. Further, by $S(r)$ we denote the Schnyder wood obtained from S by changing the orientation of the sign-separating edges regarding the solution of $A_S(r)x = \mathbf{e_1}$.

Algorithm 1 Calculation of a right triangle contact representation

Input: a 4-connected triangulation G and an aspect ratio vector $\tilde{r} \in \mathbb{R}^{n+3}_{>0}$
Output: a right triangle contact representation of G with aspect ratio vector $\tilde{r}$

```
S ← arbitrary Schnyder Wood of G
T_0 ← arbitrary right triangle contact representation of G with Schnyder wood S
r_0 ← aspect ratio vector of T_0
while x(S, r̃) ≱ 0 do
    r_1 ← r̃
    r_m ← (r_0 + r_1)/2
    while S(r_m) = S or x(S(r_m), r_m) ≱ 0 do
        if S(r_m) = S then
            r_0 ← r_m
        else
            r_1 ← r_m
        end if
        r_m ← (r_0 + r_1)/2
    end while
    S ← S(r_m)
    r_0 ← r_m
end while
calculate from x(S, r̃) a right triangle contact representation T of G
return T
```

If we assume that on the line segment we never run into an aspect ratio vector such that in the corresponding contact representation more than one face is degenerate, we can deduce the following from Lemma 5: If we know a Schnyder wood S with $r_t \in \mathcal{R}_S$, then either $\tilde{r} \in \mathcal{R}_S$ or there is a $t < t' < 1$ such that $S(r_{t'}) \neq S$ and $r_{t'} \in \mathcal{R}_{S(r_{t'})}$ (we could even assume that S and $S(r_{t'})$ are neighboring). This is exactly the idea we realize in Algorithm 1.

We cannot be sure that the inner loop always terminates because we cannot apply Lemma 5 if there are aspect ratio vectors on the line segment such that in the corresponding contact representation more than one face is degenerate. But if the inner loop always terminates, the outer loop terminates after $\mathcal{O}(n^2L)$ iterations where L is the number of Schnyder woods of G (see Section 5). Since the number of Schnyder woods can be exponential in n [4], this yields an exponential running time.

We will conclude by stating some conjectures concerning the computation of triangle contact representations. The strong experimental results we mentioned in the end of Section 3, give rise to the following conjecture:

Conjecture 2. The variant of the heuristic by Felsner we described in the end of Section 3 terminates for every planar triangulation G, every aspect ratio vector $\tilde{r}$, and every Schnyder wood S of G to start with.

Since the number of iterations has always been small in the experiments, we conjecture that it can be bounded by a polynomial in n. This would yield an algorithm with polynomial running time. Therefore we also conjecture the following:

Conjecture 3. A right triangle contact representation of a planar triangulation G with given aspect ratio vector $\tilde{r}$ can be computed in polynomial time.

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