Homothetic triangle contact representations

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Abstract. We prove that every 4-connected planar triangulation admits a contact representation by homothetic triangles.
There is a known proof of this result that is based on the Convex Packing Theorem by Schramm, a general result about contact representations of planar triangulations by convex shapes. But our approach makes use of the combinatorial structure of triangle contact representations in terms of Schnyder Woods. We start with an arbitrary Schnyder Wood and produce a sequence of Schnyder Woods via face flips. We show that at some point the sequence has to reach a Schnyder Wood describing a representation by homothetic triangles.

1 Introduction

A *triangle contact system* $\mathcal{T}$ is a finite system of triangles in the plane such that any two triangles intersect in at most one point. Moreover such an intersection point has to be a corner of exactly one of the two involved triangles. We define $G^*(\mathcal{T})$ as the graph that has a vertex for every triangle of $\mathcal{T}$ and for every triangle contact an edge between the involved triangles. For a given graph G and a triangle contact system $\mathcal{T}$ with $G^*(\mathcal{T}) = G$ we say that $\mathcal{T}$ is a *triangle contact representation* of G.

The main goal of this paper will be to prove the following result.

Theorem 1 ([5]). *Let G be a 4-connected planar triangulation. Then there is a triangle contact representation of G by homothetic triangles.*

The original proof of Theorem 1 in [5][1] makes use of the following theorem by Schramm.

Theorem 2 (Convex Packing Theorem [10]). *Let G be a triangulation with outer face $\{a, b, c\}$. Further let C be a simple closed curve in the plane partitioned into arcs $\mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c$ and for each interior vertex v of G let $\mathcal{Q}_v$ be a convex set in the plane containing more than one point. Then there is a contact representation of a supergraph of G (on the same vertex set, but possibly with more edges) where each interior vertex v is represented by a homothetic copy of its prototype $\mathcal{Q}_v$ and each outer vertex w by the arc P_w.*

[1] The journal version [6] does not contain this proof.

But if we want to calculate a triangle contact representation of G efficiently, this theorem does not help since it is purely existential. On the other hand, we know a combinatorial heuristic by Felsner [3] that calculates triangle contact representations quite fast in practical experiments [8]. However, to the best of our knowledge, it is not known whether this heuristic terminates for every instance, nor whether it has a good (e.g., polynomial) running time if it terminates.

Our new proof of Theorem 1 is based on the ideas of this heuristic. Therefore it might help to better understand this heuristic in the future. Since it makes use of the combinatorial structure of triangle contact representations, it also helps to better understand triangle contact representations themselves. Felsner and Francis [4] even explicitly ask for a proof of Theorem 1 by this approach.

In the author's master's thesis [12] with a similar approach also the existence of contact representations of 5-connected plane graphs by homothetic squares has been proved. But in that case there are other known proofs which are not based on the Convex Packing Theorem [7, 11]. That is why we will focus on contact representations by homothetic triangles.

Let us get back to triangle contact representations. In the case that $G^*(\mathcal{T})$ is a triangulation, in $\mathcal{T}$ the inner (i.e., bounded) faces of $G^*(\mathcal{T})$ are also represented by triangles. We denote these by *dual triangles* and for clear distinction the triangles of $\mathcal{T}$ by *primal triangles*.

In Theorem 1 we do not specify what is the shape of the homothetic triangles. The reason is that if we are given a contact representation by homothetic triangles, we can change the shapes of these triangles to homothetic copies of an arbitrary given triangle by a linear transformation of the plane. So we choose to prove the existence of a contact representation by right, isosceles triangles with a horizontal edge at the bottom and a vertical edge at the right hand side. We will even consider a larger class of triangle contact representations. A *right triangle contact representation* is a triangle contact representation by right triangles with a horizontal edge at the bottom and a vertical edge at the right hand side. The *aspect ratio* of such a triangle is the quotient of the lengths of its vertical and its horizontal edge. And the *aspect ratio vector* of a right triangle contact representation is the vector of the aspect ratios of its triangles (we assume the vertices of G have a fixed numbering $1, \dots, n+3$). See Fig. 1 for an example of a right triangle contact representation.

Now we can formulate a stronger theorem that implies Theorem 1.

Theorem 3. *Let G be a 4-connected triangulation and $\tilde{r} \in \mathbb{R}^{n+3}_{>0}$. Then there is a right triangle contact representation of G with aspect ratio vector $\tilde{r}$.*

The paper is organized as follows: In Section 2 we define *Schnyder Woods* as the combinatorial structure describing triangle contact representations and recall some known results about them. In Section 3 we describe a system of linear equations for calculating a right triangle contact representation with given Schnyder Wood and given aspect ratio vector. Then in Section 4 we present the heuristic for calculating right triangle contact representations we mentioned before. Finally in Section 5 we prove some lemmata we need for the proof of Theorem 3 in Section 6.

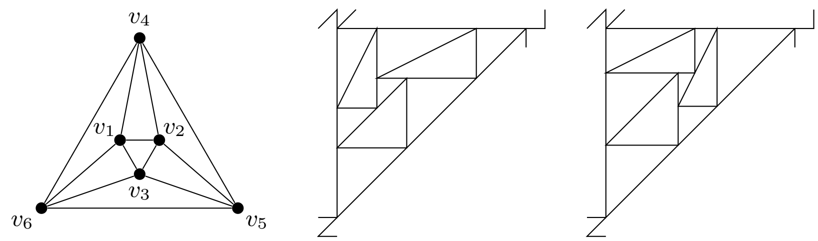


Fig. 1: Two right triangle contact representations of the same graph with aspect ratio vectors $(2, 1/2, 1, 1, 1, 1)$ and $(1/2, 2, 1, 1, 1, 1)$

Sections 2 to 4 can be seen as background literature. Our main contribution are the results in Sections 5 and 6.

2 Schnyder Woods

Schnyder Woods are a combinatorial structure on triangulations that play a central role in this paper. They were first introduced by Schnyder [9] under the name of *realizers*.

Definition 1. *Let G be a triangulation with outer vertices $v_{n+1}, v_{n+2}, v_{n+3}$ in clockwise order. Then a* Schnyder Wood *of G is an orientation and coloring of the interior edges of G with the colors red, green and blue such that*

- *each edge incident to v_{n+1} is red and incoming, each edge incident to v_{n+2} is green and incoming, and each edge incident to v_{n+3} is blue and incoming,*
- *each inner vertex has in clockwise order exactly one red, green and blue outgoing edge, and in the interval between two outgoing edges there are only incoming edges in the third color (see Fig. 2).*

If we forget about the colors of a Schnyder Wood, we obtain a 3-orientation, i.e., each inner vertex has outdegree 3 and the outer vertices have outdegree 0. The converse also holds:

Proposition 1 (de Fraysseix and Ossona de Mendez [13]). *If the graph G is a 3-orientation of a triangulation with outer vertices labeled $v_{n+1}, v_{n+2}, v_{n+3}$ in clockwise order, then there is a unique way of coloring the interior edges of G to receive a Schnyder Wood.*

Let $\mathcal{T}$ be a triangle contact system such that $G := G^*(\mathcal{T})$ is a triangulation. If we orient each inner edge of G from the triangle whose corner is involved in the contact to the other triangle, we obtain an orientation where the outdegree of each inner vertex is at most 3. As a consequence of Euler's formula, G has exactly $3n$ inner edges, and therefore the outdegree of each inner vertex has to be exactly 3. Thus $\mathcal{T}$ induces a 3-orientation and hence a Schnyder Wood of G (we will call this Schnyder Wood the Schnyder Wood of $\mathcal{T}$). Again the converse also holds:

Proposition 2 (de Fraysseix, Ossona de Mendez and Rosenstiehl [14]). *Let G be a triangulation and S a Schnyder Wood of G. Then there is a right triangle contact representation of G with induced Schnyder Wood S.*

For right triangle contact representations we can obtain the colors of the associated Schnyder Wood also directly, without using Proposition 1. We color an edge red if it corresponds to the upper corner of a triangle, green if it corresponds to the right lower corner of a triangle, and blue if it corresponds to the left lower corner of a triangle. See Fig. 3 for an example.

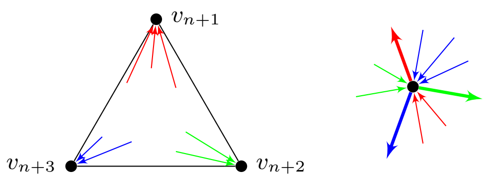


Fig. 2: The local conditions of a Schnyder Wood

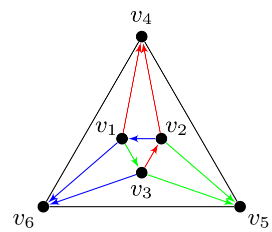


Fig. 3: The Schnyder Wood induced by the first example of Fig. 1

3 The System of Linear Equations

For the whole section let G be a triangulation with inner vertices $v_1, \dots, v_n$ and outer vertices $v_{n+1}, v_{n+2}, v_{n+3}$ in clockwise order, let S be a Schnyder Wood of G and let $r \in \mathbb{R}^{n+3}_{>0}$ be an aspect ratio vector for G. We will now describe a system of linear equations for calculating the edge lengths of a right triangle contact representation of G with aspect ratio vector r and induced Schnyder Wood S. This system has been introduced by Felsner [3] and studied by Rucker [8] for the special case $r = (1, \dots, 1)$. All results in this section are due to them.

For each inner vertex v of G we have a variable x_v which represents the width of the corresponding primal triangle, and for each inner face f of G a variable x_f which represents the width of the corresponding dual triangle. For a vertex v of G we denote by $\delta_r(v)$ the set of incident faces of v which are located in the interval between the green and blue outgoing edge of v. Analogously we define the sets $\delta_g(v)$ and $\delta_b(v)$. In a right triangle contact representation a primal triangle T hands down his aspect ratio to each dual triangle whose hypotenuse is contained in the hypotenuse of T. If T corresponds to the vertex v of G, these are exactly the dual triangles corresponding to the faces in $\delta_g(v)$. Therefore, if we are given the Schnyder Wood and the aspect ratio vector of a right triangle contact representation, we are implicitly also given the aspect ratios of the dual triangles. We denote the aspect ratio of a dual triangle corresponding to the face f of G by r_f. Now we can write down the equation system:

$$\sum_{f \in \delta_r(v_{n+1})} x_f = 1 \ , \tag{1}$$

$$\sum_{f \in \delta_r(v_i)} x_f - x_{v_i} = 0 \ , \quad i = 1, \dots, n \ , \tag{2}$$

$$\sum_{f \in \delta_g(v_i)} x_f - x_{v_i} = 0 \ , \quad i = 1, \dots, n \ , \tag{3}$$

$$\sum_{f \in \delta_b(v_i)} r_f x_f - r_{v_i} x_{v_i} = 0 \ , \quad i = 1, \dots, n \ . \tag{4}$$

Equation (2) says that the length of the horizontal edge of a primal triangle is equal to the sum of the lengths of the adjacent dual triangles. Equations (3) and (4) say the same for the other two edges of a primal triangle. And the purpose of (1) is to pick one single solution out of the space of solutions of the apart from that homogeneous equation system. We will also use the shorter notation $A_S(r)x = \mathbf{e_1}$ for the equation system.

Proposition 3. *The system $A_S(r)x = \mathbf{e_1}$ is uniquely solvable.*

Proof. See Appendix A. □

Because of the way we chose the equations of the system, it is clear that the existence of a right triangle contact representation of G with Schnyder Wood S and aspect ratio vector r implies a nonnegative solution. The following proposition shows that also the converse holds.

Proposition 4. *Let $A_S(r)x = \mathbf{e_1}$. There is a right triangle contact representation of G with induced Schnyder Wood S and aspect ratio vector r if and only if $x \geq 0$. If $x \geq 0$, the representation is unique inside the three outer triangles.*

Proof. Suppose $x \geq 0$. Let the *skeleton graph* G_{skel} of G be defined as the medial graph of G. We color and orient the edges of G_{skel} according to the rules of Fig. 4 in dependence of S. Then a right triangle contact representation of G with Schnyder Wood S is a rectilinear embedding of G_{skel} with horizontal red edges oriented to the right and vertical blue edges oriented to the top (see Fig. 5). The solution vector x gives us the two-dimensional lengths of these edges by

$$\ell(e) := \begin{cases} (x_f, 0) \ , & \text{if } e \text{ is red,} \\ (0, r_f x_f) \ , & \text{if } e \text{ is blue,} \\ (x_f, r_f x_f) \ , & \text{if } e \text{ is green.} \end{cases}$$

Along the lines of the proof of [2, Theorem 2.3] it can be shown that this indeed leads to a proper triangle contact representation. See Fig. 5 for an example. □

Next we will prove a result about the structure of nonnegative solutions with zero entries.

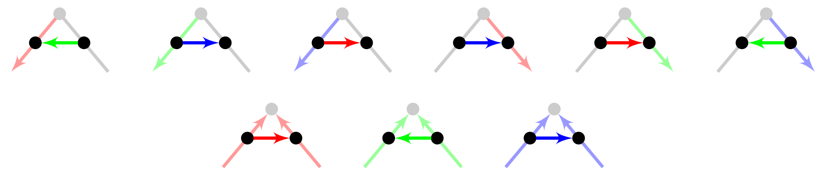
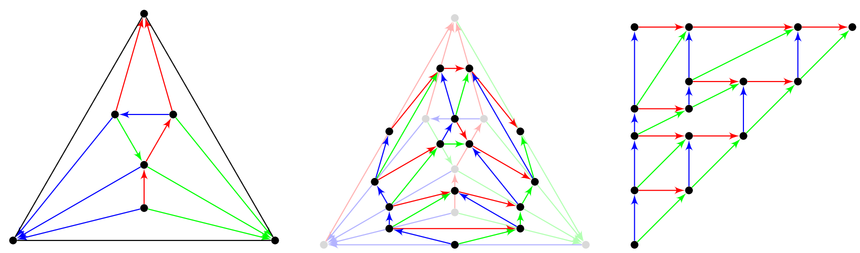


Fig. 4: Coloring and orientation rules for G_{skel}. The gray vertices are the vertices of G and the black vertices are the vertices of G_{skel}.



(a) G with Schnyder Wood (b) G_{skel} (c) Embedding of G_{skel}

Fig. 5: Construction of a right triangle contact representation from a nonnegative solution of $A_S(r)x = \mathbf{e_1}$

Definition 2. *Let $A_S(r)x = b$ and $x \geq 0$. Then an edge e of G is called a* transition edge *if it is incident to inner faces f_1 and f_2 of G with $x_{f_1} > 0$ and $x_{f_2} = 0$.*

Lemma 1. *Let $A_S(r)x = b$ and $x \geq 0$. Then the transition edges of G form an edge disjoint union of cycles of length 3. Moreover there is no edge going from a vertex on such a cycle into the interior of this cycle.*

Proof. For each transition edge e we denote the successor edge by $\phi(e)$ (this will be the successor of e on the cycle of e). Now let e be a transition edge that is outgoing for vertex v. Since there are no negative faces and v is incident to at least one positive face, we have $x_v > 0$ Let $e_1 := e, e_2, e_3, \ldots$ be the cyclic order of edges incident to v such that the face f_1 between e_1 and e_2 fulfills $x_{f_1} = 0$. Because of $x_v > 0$ there has to be an edge e_k such that each edge $e_2, \ldots, e_k$ is oriented towards v, and for $i = 1, \ldots, k-1$, the face f_i between e_i and e_{i+1} fulfills $x_{f_i} = 0$, and the face f_k between e_k and e_{k+1} fulfills $x_{f_k} > 0$. Then we set $\phi(e) := e_k$.

Now we view ϕ as a map from the set of transition edges into the set of transition edges. It is obvious that $|\phi^{-1}(e)| \leq 1$ for every transition edge e. Since the domain and codomain of ϕ are the same, this implies that ϕ is a bijection. Therefore ϕ defines a partition of these edges into edge-disjoint cycles.

Let C be one of these cycles and let us assume C is simple. Furthermore let k be the length of C, let n_v be the number of vertices and n_e the number of edges in the interior of C. Then we get by Euler's formula $n_e = 3n_v + k - 3$.

There are $2k$ outgoing edges of vertices on C not belonging to C. Because of the construction of C, they are all pointing into the interior of C or all pointing into the exterior of C. In the first case we have $n_e = 3n_v + 2k$, contradicting $k \geq 0$, and in the second case we have $n_e = 3n_v$, what implies $k = 3$.

That C has to be simple can be seen similarly by double counting the edges inside a subcycle of C. □

4 A heuristic

We now present a heuristic by Felsner [3] for calculating right triangle contact representations that is based on the linear equation system of Section 3.

In Definition 2 we have defined transition edges for the case $x \geq 0$. In the case $x \ngeq 0$ we call an edge of G a transition edge if it is incident to inner faces f_1 and f_2 of G with $x_{f_1} \geq 0$ and $x_{f_2} < 0$. Analogously to Lemma 1 it can be shown that these edges again form a edge-disjoint union of directed cycles.

Algorithm 1 Calculation of a right triangle contact representation

Input: a 4-connected triangulation G and an aspect ratio vector $\tilde{r} \in \mathbb{R}^{n+3}_{>0}$
Output: a right triangle contact representation of G with aspect ratio vector $\tilde{r}$

```
S ← arbitrary Schnyder Wood of G
x ← unique solution of A_S(r̃)x = e1
while x ≱ 0 do
    change Schnyder Wood S by reversing all transition edges
    x ← unique solution of A_S(r̃)x = e1
end while
calculate from x a right triangle contact representation T of G
return T
```

In experiments by Rucker [8] this heuristic delivered good results, i.e., it always terminated after a small number of iterations. But in theory we neither know whether it terminates for every instance, nor know any nontrivial bounds for the number of iterations it takes in the case of termination. Also variants of this heuristic have been studied where the Schnyder Wood is changed in some other way, but without success.

5 Preparation of the proof

5.1 Feasible aspect ratio vectors for a fixed Schnyder Wood

We call a triangle contact representation *degenerate* if the corresponding solution vector of the equation system has zero entries. Otherwise we call it *nondegenerate*. For a fixed Schnyder Wood S we denote the set of aspect ratio vectors of nondegenerate right triangle contact representations of G by $\mathcal{R}_S$ and the set of aspect ratio vectors of all (possibly degenerate) triangle contact representations of G by $\bar{\mathcal{R}}_S$.

In Proposition 4 we have seen that $r \in \mathcal{R}_S$ if and only if $x > 0$ where x is the solution of $A_S(r)x = \mathbf{e_1}$. With Cramer's rule we then get the following:

Lemma 2. *There are polynomials $p_1, \dots, p_{3n+1}$ in the variables $r_1, \dots, r_{n+3}$ with $\deg p_j \leq 3n+1$ for each j such that*

$$\mathcal{R}_S = \{r \in \mathbb{R}^{n+3}_{>0} : p_j(r) > 0 \text{ for } j = 1, \dots, 3n+1\} ,$$
$$\bar{\mathcal{R}}_S = \{r \in \mathbb{R}^{n+3}_{>0} : p_j(r) \geq 0 \text{ for } j = 1, \dots, 3n+1\} .$$

In particular $\mathcal{R}_S$ is an open and $\bar{\mathcal{R}}_S$ a closed set.

The following lemma is a corollary of Lemma 2 and shows that the intersection of $\mathcal{R}_S$ with a line segment decomposes into a bounded number of intervals.

Lemma 3. *Let $r_0, r_1 \in \mathbb{R}^{n+3}_{>0}$ be two distinct aspect ratio vectors and for each $0 \leq t \leq 1$ let $r_t := (1-t)r_0 + tr_1$. Then there are open intervals $I_1, \dots, I_k$ with $k \leq (3n+1)\lfloor \frac{3n+1}{2} \rfloor + 1$ such that*

$$I_1 \cup \dots \cup I_k = \{t \in \mathbb{R} : 0 < t < 1, r_t \in \mathcal{R}_S\} ,$$
$$\bar{I}_1 \cup \dots \cup \bar{I}_k \subseteq \{t \in \mathbb{R} : 0 \leq t \leq 1, r_t \in \bar{\mathcal{R}}_S\} .$$

5.2 Neighboring Schnyder Woods

We call two Schnyder Woods S and S' *neighboring* if the corresponding 3-orientations differ in a single 3-cycle C. In this case we call C the *difference cycle* of S and S'. The set of Schnyder Woods of a fixed graph has been thoroughly studied and it is well known that it has the structure of a distributive lattice with the cover relation being exactly this neighboring relation [1].

The fact that contact representations of 5-connected triangulations by homothetic squares are unique [7, 11], gives rise to the following conjecture.

Conjecture 1. Let G be a 4-connected triangulation and $\tilde{r} \in \mathbb{R}^{n+3}_{>0}$. Then the right triangle contact representation of G with aspect ratio vector $\tilde{r}$ is unique.

The following proposition can be seen as a weak variant of this statement.

Proposition 5. *Let S and S' be two neighboring Schnyder Woods. Furthermore let $r \in \mathcal{R}_S$. Then $r \notin \bar{\mathcal{R}}_{S'}$.*

Proof. See Appendix A. □

Later our strategy will be to move along a line segment of aspect ratio vectors somehow keeping the invariant that there exits a right triangle contact representation with the current aspect ratio vector. In this process, the Schnyder Wood of this triangle contact representation will stay the same for a whole subsegment of this line segment. The following lemma allows us to switch to a neighboring Schnyder Wood if the current one does not work any more.

Lemma 4. *Let* $\{s_t : 0 \leq t \leq 1\}$ *be a line segment of aspect ratio vectors. If there is a* $0 < t_0 < 1$ *such that* $s_{t_0} \in \bar{\mathcal{R}}_S \setminus \mathcal{R}_S$ *and an* $\varepsilon > 0$ *such that for each* $t_0 - \varepsilon \leq t < t_0$ *we have* $s_t \in \mathcal{R}_S$*, then for each* $\varepsilon' > 0$ *there is an aspect ratio vector* $r \in B(s_{t_0}, \varepsilon')$ *and a neighboring Schnyder Wood* S' *of* S *with* $r \in \mathcal{R}_{S'}$*.*

We will now give a sketch of a proof of this lemma. The full proof can be found in Appendix B.

From Lemma 1 we know that in the right triangle contact representation with Schnyder Wood S and aspect ratio vector s_{t_0} only some dual triangles are degenerate (let $f_1, \ldots, f_k$ be the corresponding faces of G), but all primal triangles are nondegenerate. The idea of the proof is to take a nondegenerate right triangle contact representation $\mathcal{T}$ of G with aspect ratio vector $s_{t_0-\delta}$ for a small $\delta > 0$ and move segments such that we first reach a representation with only one single face f_i degenerate and then a representation where f_i is flipped. Since in $\mathcal{T}$ the face f_i is small compared to the sizes of the primal triangles, small segment movements suffice, and therefore the aspect ratio vector doesn't change much.

6 Proof of Theorem 3

We assume there is no right triangle contact representation of G with aspect ratio vector $\tilde{r}$. The idea of the following proof is to construct under this assumption a line segment contradicting Lemma 3. For that we will construct an infinite sequence $(S_i)_{i\geq 0}$ of Schnyder Woods, two sequences (r_i) and (r'_i) of aspect ratio vectors and two sequences (ε_i) and (ε'_i) of positive real numbers fulfilling the following invariants:

(I1) For each $r \in B(r_i, \varepsilon_i)$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector r and Schnyder Wood S_i.
(I2) For each $r' \in B(r'_i, \varepsilon'_i)$ the line segment $\{(1-t)r' + t\tilde{r} : 0 \leq t \leq 1\}$ intersects the balls $B(r_0, \varepsilon_0), \ldots, B(r_i, \varepsilon_i)$ in this order (with increasing t).
(I3) The points r'_i, r_i and $\tilde{r}$ are collinear and aligned in this order.
(I4) The Schnyder Woods S_i and S_{i+1} are neighboring.

It now remains to show how to construct these sequences and why the existence of these sequences contradicts Lemma 3.

6.1 Construction of the sequences

Let S_0 be an arbitrary Schnyder Wood of G, let $\mathcal{T}_0$ be an arbitrary nondegenerate right triangle contact representation of G with Schnyder Wood S_0 and let r_0 be the aspect ratio vector of $\mathcal{T}_0$. Then we know from Lemma 2 that there is a $0 < \varepsilon_0 < 1$ such that for each $r \in B(r_0, \varepsilon_0)$ a nondegenerate right triangle contact representation of G with aspect ratio vector r and Schnyder Wood S_0 exists. Furthermore we set $r'_0 := r_0$ and $\varepsilon'_0 := \varepsilon_0$. These initial values obviously fulfill each of the four invariants.

Now we describe how to construct the $(j+1)$th sequence members from the jth ones. We set $s_t := (1-t)r'_j + t\tilde{r}$ for $0 \le t \le 1$. Then because of (I3) there is a $0 \le \hat{t} < 1$ with $s_{\hat{t}} = r_j$. Thus Lemma 3 gives us a $\delta > 0$ such that for each $\hat{t} \le t < \hat{t} + \delta$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector s_t and Schnyder Wood S_j and a degenerate one with aspect ratio vector $s_{\hat{t}+\delta}$ and Schnyder Wood S_j. Because of our assumption we have $\hat{t}+\delta < 1$ and because of $s_{\hat{t}+\delta} \notin B(r_j, \varepsilon_j)$ we have $\|r_j - s_{\hat{t}+\delta}\| > \varepsilon_j$. Now we set

$$\delta' := \min\left\{\left(1-(\hat{t}+\delta)\right)\varepsilon'_j, \|r_j - s_{\hat{t}+\delta}\| - \varepsilon_j\right\} > 0 \ .$$

Then Lemma 4 gives us an $r_{j+1} \in B^\circ(s_{\hat{t}+\delta}, \delta')$ such that there is a nondegenerate right triangle contact representation of G with aspect ratio vector r_{j+1} and a neighboring Schnyder Wood S_{j+1} of S_j. Now we set $r'_{j+1} := r'_j + \frac{1}{1-(\hat{t}+\delta)}(r_{j+1} - s_{\hat{t}+\delta})$. Then

$$\|r'_{j+1} - r'_j\| = \frac{1}{1-(\hat{t}+\delta)}\|r_{j+1} - s_{\hat{t}+\delta}\| < \frac{1}{1-(\hat{t}+\delta)}\delta' \le \varepsilon'_j$$

and therefore we have $r'_{j+1} \in B^\circ(r'_j, \varepsilon'_j)$. Moreover Lemma 2 gives us an $0 < \varepsilon_{j+1} < \delta' - \|r_{j+1} - s_{\hat{t}+\delta}\|$ such that for each $r \in B(r_{j+1}, \varepsilon_{j+1})$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector r and Schnyder Wood S_{j+1}. Finally we set $\varepsilon'_{j+1} := \frac{1}{1-(\hat{t}+\delta)}\varepsilon_{j+1}$. See Fig. 6 for an illustration of the construction.

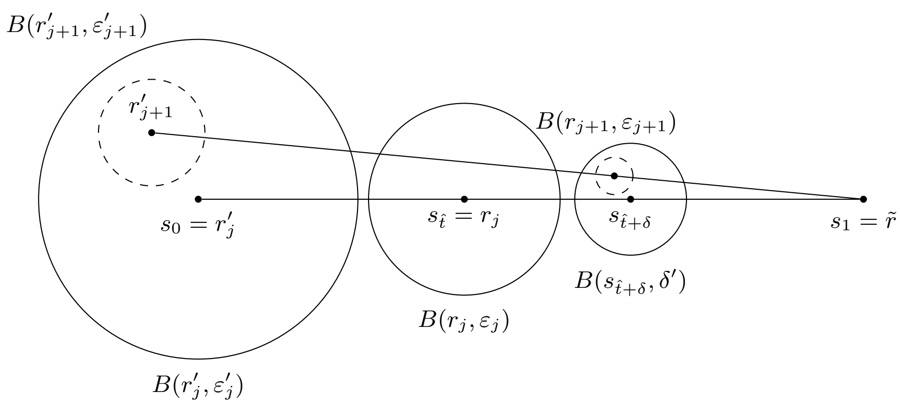


Fig. 6: The construction of the new sequence members

Clearly the invariants (I1), (I3) and (I4) are fulfilled again. Because of $B(r'_{j+1}, \varepsilon'_{j+1}) \subseteq B(r'_j, \varepsilon'_j)$ for each $r' \in B(r'_{j+1}, \varepsilon'_{j+1})$ the line segment $\{(1-t)r' + t\tilde{r} : 0 \le t \le 1\}$ intersects the balls $B(r_0, \varepsilon_0), \ldots, B(r_j, \varepsilon_j)$ in the right order. From the construction it immediately follows that the ball $B(r_{j+1}, \varepsilon_{j+1})$ is intersected by this line segment, too. And because of $\delta' \leq \|r_j - s_{\hat{t}+\delta}\| - \varepsilon_j$ and $B(r_{j+1}, \varepsilon_{j+1}) \subseteq B^\circ(s_{\hat{t}+\delta}, \delta')$ the intersection point with $B(r_{j+1}, \varepsilon_{j+1})$ is closer to $\tilde{r}$ than the intersection point with $B(r_j, \varepsilon_j)$. Therefore also (I2) is fulfilled again.

6.2 Producing a contradiction

Let L be the number of Schnyder Woods of G and $c := (3n+1)\lfloor \frac{3n+1}{2} \rfloor + 1$. We set $K := Lc + 1$. Then it follows from the pigeonhole principle that there is a Schnyder Wood S such that there are indices $0 \leq i_1 < \cdots < i_{c+1} \leq K$ with $S := S_{i_1} = \cdots = S_{i_{c+1}}$. For $l = 1, \ldots, c+1$ let $\hat{r}_l$ be an intersection point of the line segment $\{(1-t)r'_K + t\tilde{r} : 0 \leq t \leq 1\}$ and the ball $B(r_{i_l}, \varepsilon_{i_l})$. Thus for $l = 1, \ldots, c+1$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector $\hat{r}_l$ and Schnyder Wood S. From Lemma 3 it follows that the intersection of $\{(1-t)r'_K + t\tilde{r} : 0 \leq t \leq 1\}$ and $\mathcal{R}_S$ is a disjoint union of at most c open intervals. Therefore there is an l such that $\hat{r}_l$ and $\hat{r}_{l+1}$ belong to the same interval. Particularly for each $0 \leq \tau \leq 1$ there is a nondegenerate right triangle contact representation of G with aspect ratio vector $(1-\tau)\hat{r}_l + \tau\hat{r}_{l+1}$ and Schnyder Wood S. Because of (I4) the Schnyder Woods $S' := S_{i_l+1}$ and S are neighboring. And because of (I2) there is a $0 \leq \tau' \leq 1$ with $(1-\tau')\hat{r}_l + \tau'\hat{r}_{l+1} \in B(r_{i_l+1}, \varepsilon_{i_l+1})$. But then because of (I1) there is also a nondegenerate right triangle contact representation of G with aspect ratio vector $(1-\tau')\hat{r}_l + \tau'\hat{r}_{l+1}$ and Schnyder Wood S', contradicting Proposition 5. □

7 Conclusion and open problems

We proved the existence of homothetic triangle contact representations of 4-connected triangulations by leveraging on the inner structure of those representations.

The main question that remains open is whether and how homothetic triangle contact representations can be computed efficiently. And related to that question we would like to know if Algorithm 1 terminates for every instance and how many iterations it takes.

Another interesting open question is whether Conjecture 1 is true, i.e., whether homothetic triangle contact representations are unique.

A Appendix: Proofs of Propositions 3 and 5

For the following proofs we need a technical lemma about perfect matchings in plane bipartite graphs. So let H be a plane bipartite graph with vertex classes $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$. Then a perfect matching of H induces a permutation $\sigma \in \mathcal{S}_k$ by $\sigma(i) = j :\Leftrightarrow \{a_i, b_j\} \in M$. We define the *sign* $\operatorname{sgn}(M)$ of a perfect matching M as the sign of the corresponding permutation.

Lemma 5. *Let each inner face f of H be bounded by a simple cycle of length $\ell_f \equiv 2 \mod 4$. Then for two perfect matchings M and M' of H we get* $\operatorname{sgn}(M) = \operatorname{sgn}(M')$.

Proof. Under the given conditions, for each cycle of length ℓ with k' vertices in its interior the formula $\ell + 2k' \equiv 2 \mod 4$ is valid. This can be proved by induction on the number of faces enclosed by the cycle.

The symmetric difference of M and M' is a disjoint union of simple cycles $C_1, \ldots, C_m$, and each of these cycles contains an even number of vertices in its interior. Thus, for $i = 1, \ldots, m$, the cycle C_i has length $4n_i + 2$ for an $n_i \in \mathbb{N}$. Therefore, on the vertices of C_i the permutation σ corresponding to M and the permutation σ' corresponding to M' differ in a cyclic permutation τ_i of length $2n_i + 1$. Hence we have $\sigma' = \sigma \circ \tau_1 \circ \cdots \circ \tau_m$ and therefore

$$\begin{aligned}\operatorname{sgn}(\sigma') &= \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau_1) \cdots \operatorname{sgn}(\tau_m) \\ &= \operatorname{sgn}(\sigma) \cdot (-1)^{2n_1} \cdots (-1)^{2n_m} = \operatorname{sgn}(\sigma) \ .\end{aligned}$$

□

Proposition 3. *The system $A_S(r)x = \mathbf{e_1}$ is uniquely solvable.*

Proof. The system consists of $3n + 1$ equations. And it can easily be seen that the number of variables is $3n + 1$, too. Thus $A_S(r)$ is a square matrix and it suffices to show that $\det(A_S(r)) \neq 0$.

From the Leibniz formula we have $\det(A_S(r)) = \sum_\sigma \operatorname{sgn}(\sigma) \prod_i (A_S(r))_{i\sigma(i)}$. For a permutation σ we set

$$\operatorname{signat}(\sigma) := \begin{cases} +1 \ , & \text{if } \operatorname{sgn}(\sigma) \prod_i (A_S(r))_{i\sigma(i)} > 0 \ , \\ -1 \ , & \text{if } \operatorname{sgn}(\sigma) \prod_i (A_S(r))_{i\sigma(i)} < 0 \ , \\ 0 & \text{else.} \end{cases}$$

We will split the proof of $\det(A_S(r)) \neq 0$ into two parts. First we show that if $\operatorname{signat}(\sigma), \operatorname{signat}(\tau) \neq 0$ for two permutations σ and τ, then $\operatorname{signat}(\sigma) = \operatorname{signat}(\tau)$. And after that we show that there exists a permutation σ with $\operatorname{signat}(\sigma) \neq 0$.

As a tool for these two proofs we now define a bipartite graph G_S with edges colored black and red. The first vertex class consists of the variables of the equation system and is denoted by $a_1, \ldots, a_k$. The second vertex class consists of the equations of the equation system and is denoted by $b_1, \ldots, b_k$. There is a black edge a_ib_j in G_S if and only if $A_S(r)_{ij} > 0$ and a red edge a_ib_j if and only if $A_S(r)_{ij} < 0$. Notice that the definition of G_S is independent of r. A triangle contact system with induced Schnyder Wood S induces a planar embedding of G_S (see Fig. 7). Therefore we will consider G_S as a plane graph.

Let σ be a permutation with $\operatorname{signat}(\sigma) \neq 0$. Then σ corresponds to a perfect matching of G_S. A perfect matching of G_S contains exactly n red edges because exactly the n variable-vertices of G_S corresponding to an inner vertex of G are incident to red edges in a perfect matching of G_S. The other variable-vertices are incident to black edges. Thus $\operatorname{signat}(\sigma) = (-1)^n \operatorname{sgn}(\sigma)$. Let τ be another permutation with $\operatorname{signat}(\tau) \neq 0$. Then also $\operatorname{signat}(\tau) = (-1)^n \operatorname{sgn}(\tau)$. Therefore it remains to show that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)$. But that immediately follows from the fact that in G_S every inner face is bounded by a circle of length $6 \equiv 2 \mod 4$ (see Lemma 5).

Now we come to the second claim. We will describe the construction of a perfect matching in G_S by giving every equation-vertex v its matching partner:

- If v corresponds to the inhomogeneous equation, it is matched with the leftmost adjacent dual vertex.
- If v corresponds to the hypotenuse of a triangle, it is matched with the adjacent primal vertex.
- If v corresponds to the horizontal side of a triangle, it is matched with the leftmost adjacent dual vertex.
- If v corresponds to the vertical side of a triangle, it is matched with the upmost adjacent dual vertex.

Figure 8 illustrates this construction. It can easily be verified that this is indeed a perfect matching, and since the existence of a perfect matching in G_S is equivalent to the existence of a permutation σ with $\operatorname{signat}(\sigma) \neq 0$, this finishes the proof. □

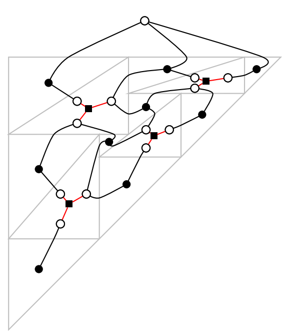


Fig. 7: Planar embedding of G_S induced by a triangle contact system

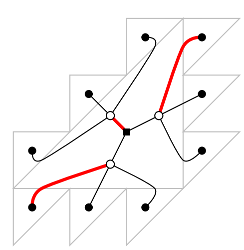
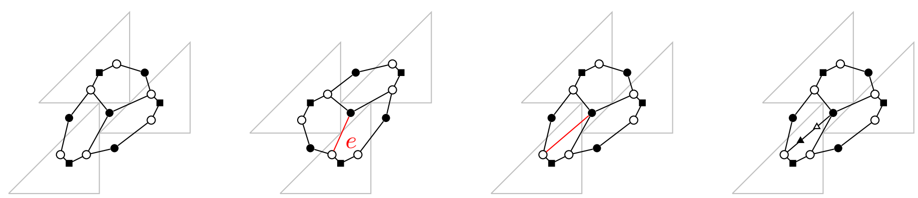


Fig. 8: Construction of a perfect matching in G_S (the matching edges of a primal triangle)

Proposition 5. *Let S and S' be two neighboring Schnyder Woods. Furthermore let $r \in \mathcal{R}_S$. Then $r \notin \bar{\mathcal{R}}_{S'}$.*

Proof. Let x and x' be the unique solutions of $A_S(r)x = \mathbf{e_1}$ and $A_{S'}(r)x' = \mathbf{e_1}$. Then we know $x > 0$ and need to show $x' \ngeq 0$ (see Proposition 4).

The first fact we want to prove is $\det(A_S(r))\det(A_{S'}(r)) < 0$. We do that by showing that any perfect matching σ of $H := G_S$ and any perfect matching σ' of $H' := G_{S'}$ have the same signature. For that let C be the difference cycle of S and S'. Since G is 4-connected, C is the boundary of a face f of G. Let e be the edge of σ' that is incident to the vertex f of H'. Then we define a new graph $\tilde{H}$ by adding the edge e to H such that σ and σ' are two perfect matchings of $\tilde{H}$. Finally we define another new graph $\hat{H}$ by subdividing e in $\tilde{H}$ into three edges. Figure 9 illustrates the graphs H, H', $\tilde{H}$ and $\hat{H}$. It can easily be seen that in $\hat{H}$ each inner face is bounded by a cycle of length $6 \equiv_4 2$.



(a) The graph H (b) The graph H' (c) The graph $\tilde{H}$ (d) The graph $\hat{H}$

Fig. 9: The auxiliary graphs in the proof of Proposition 5

Let D be the symmetric difference of σ and σ'. Then D is a disjoint union of cycles $\tilde{C}_1, \ldots, \tilde{C}_k$ in $\tilde{H}$ where w.l.o.g. $\tilde{C}_1$ contains the edge e. Let $\hat{C}_1, \ldots, \hat{C}_k$ be the corresponding cycles in $\hat{H}$. Then the lengths of the cycles fulfill $\ell(\hat{C}_1) = \ell(\tilde{C}_1) + 2$ and $\ell(\hat{C}_i) = \ell(\tilde{C}_i)$ for $i = 2, \ldots, k$. Thus $\ell(\tilde{C}_1) \equiv_4 0$ and $\ell(\tilde{C}_i) \equiv_4 2$ for $i = 2, \ldots, k$. Therefore

$$\operatorname{sgn}(\sigma') = \operatorname{sgn}(\sigma) \prod_{i=1}^{k} (-1)^{\frac{1}{2}\ell(\tilde{C}_i)-1} = -\operatorname{sgn}(\sigma) \ .$$

Thus we have $\operatorname{signat}(\sigma') = -\operatorname{signat}(\sigma)$ what implies $\det(A_S(r))\det(A_{S'}(r)) < 0$.

The second fact we now want to prove is $\det(A_S^{(j)}(r))\det(A_{S'}^{(j)}(r)) > 0$ where j is the index of the column corresponding to f. But that's clear since $A_S(r)$ and $A_{S'}(r)$ only differ in the jth column such that $A_S^{(j)}(r) = A_{S'}^{(j)}(r)$.

Putting $\det(A_S(r))\det(A_{S'}(r)) < 0$, $\det(A_S^{(j)}(r))\det(A_{S'}^{(j)}(r)) > 0$ and $x_j > 0$ together we finally get $x'_j < 0$. □

B Appendix: Proof of Lemma 4

Lemma 4. *Let $\{s_t : 0 \leq t \leq 1\}$ be a line segment of aspect ratio vectors. If there is a $0 < t_0 < 1$ such that $s_{t_0} \in \bar{\mathcal{R}}_S \setminus \mathcal{R}_S$ and an $\varepsilon > 0$ such that for each $t_0 - \varepsilon \leq t < t_0$ we have $s_t \in \mathcal{R}_S$, then for each $\varepsilon' > 0$ there is an aspect ratio vector $r \in B(s_{t_0}, \varepsilon')$ and a neighboring Schnyder Wood S' of S with $r \in \mathcal{R}_{S'}$.*

Proof. We first observe that we can reconstruct a right triangle contact representation if we are given the three outer triangles and the y-coordinates of the horizontal edges (we will call them *base coordinates*) of all inner primal triangles. For that we construct the inner triangles in the order of a canonical ordering, i.e., a topological sorting of $T_r \cup {T_g}^{-1} \cup {T_b}^{-1}$ where T_r is the set of red edges, T_g the set of green edges and T_b the set of blue edges in the Schnyder Wood S [14].

For $t_0 - \varepsilon \leq t \leq t_0$ let $\mathcal{T}_t$ be the right triangle contact representation of G with Schnyder Wood S and aspect ratio vector s_t. Then in $\mathcal{T}_{t_0}$ some dual triangles are degenerate. Let T_f be a leftmost degenerate dual triangle in $\mathcal{T}_{t_0}$ and f the corresponding face of G. As a first step we will now construct a right triangle contact representation $\mathcal{T}'$ with only T_f degenerate.

Let $h_{\min}, h_{\max}, w_{\min}, w_{\max} > 0$ be the minimal height, maximal height, minimal width and maximal width of all primal triangles in all $\mathcal{T}_t$, $t_0 - \varepsilon \leq t \leq t_0$. Furthermore let $\tilde{h}_{\min} > 0$ be the minimal height of all dual triangles, that are not degenerate in $\mathcal{T}_{t_0}$, in all $\mathcal{T}_t$, $t_0 - \varepsilon \leq t \leq t_0$. We set

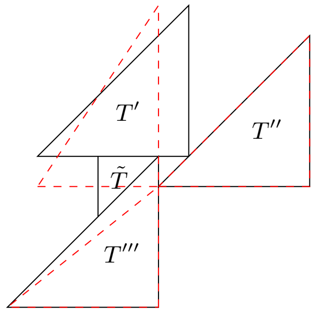
$$\delta := \min\left\{\frac{\varepsilon' w_{\min}^2}{6nh_{\max} + 3w_{\max} + 2n\varepsilon' w_{\min}}, \tilde{h}_{\min}\right\} > 0 \ .$$

Then there is a $t_0 - \varepsilon < t' < t_0$ such that in $\mathcal{T}_{t'}$ the height of T_f is smaller than $\min\{\delta, \frac{h_{\min}}{w_{\max}}\delta\}$ and $\|s_{t_0} - s_{t'}\|_\infty \leq \frac{\varepsilon'}{3}$.

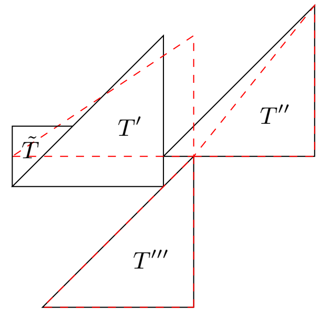
Let T', T'', T''' be the adjacent primal triangles of T_f and let furthermore $\tilde{T}$ be the dual triangle as in Fig. 10. Then we define the right triangle contact representation $\mathcal{T}'$ by changing the base coordinate of T' in $\mathcal{T}_{t'}$ to the value of the base coordinate of T''. Since the difference Δ_y of the base coordinates of T' in $\mathcal{T}_{t'}$ and $\mathcal{T}'$ fulfills $\Delta_y < \delta \leq \tilde{h}_{\min} \leq h(\tilde{T})$, where $h(\tilde{T})$ is the height of $\tilde{T}$ in $\mathcal{T}_{t'}$, the new base coordinates indeed allow us to construct a right triangle representation with only T_f degenerate.

Let $T_1, \ldots, T_n$ be the inner primal triangles in the construction order (a canonical ordering). We will now prove by induction on k that the difference $\Delta_x^{(k)}$ of the x-coordinates of the right segments of T_k in $\mathcal{T}_{t'}$ and $\mathcal{T}'$ is at most $k\delta$. For that let $k \in \{1, \ldots, n\}$ and let v be the vertex of G corresponding to T_k. We also write T_v for T_k. Let (v, u) be the green outgoing edge of v and T_u the primal triangle corresponding to u. If $T_v = T'$, then $\Delta_x^{(k)}$ is the width of T_f in $\mathcal{T}_{t'}$ and thus $\Delta_x^{(k)} < \delta \leq k\delta$. Otherwise the base coordinate of T_v doesn't change. In the iteration when T_v is being constructed in $\mathcal{T}'$, the triangle T_u has already been constructed. We distinguish several cases concerning T_u.

Case 1: The y-coordinates of the left lower corner and the upper corner of T_u haven't changed. From induction hypothesis the x-coordinates of these corners



(a) C clockwise



(b) C counterclockwise

Fig. 10: The construction of $\mathcal{T}'$ (dashed, red) from $\mathcal{T}_{t'}$ (solid, black) for the two possible orientations of the boundary cycle C of f

have changed at most $(k-1)\delta$. But this immediately implies $\Delta_x^{(k)} \leq (k-1)\delta \leq k\delta$ (see Fig. 11).

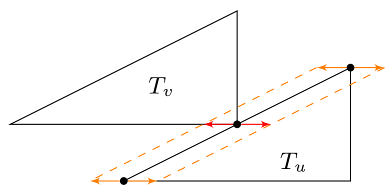


Fig. 11: Change of the x-coordinate of the right segment of T_v (case 1)

Case 2: The y-coordinate of the upper corner of T_u has changed. Then the change of this y-coordinate is at most $\frac{h_{\min}}{w_{\max}}\delta$. And the y-coordinate of the left lower corner of T_u hasn't changed. Let h and w be the height and width of T_u in $\mathcal{T}_{t'}$ and let d be defined as in Fig. 12. Then $d \leq \frac{w}{h}\frac{h_{\min}}{w_{\max}}\delta \leq \delta$. Therefore it follows like in case 1 that $\Delta_x^{(k)} \leq d + (k-1)\delta \leq k\delta$.

Case 3: The y-coordinate of the left lower corner of T_u has changed. Then the change of this y-coordinate is at most $\frac{h_{\min}}{w_{\max}}\delta$ and the y-coordinate of the upper corner of T_u hasn't changed. Let h and w be the height and width of T_u in $\mathcal{T}_{t'}$ and let d be defined as in Fig. 13. Then again $d \leq \frac{w}{h}\frac{h_{\min}}{w_{\max}}\delta \leq \delta$ and therefore $\Delta_x^{(k)} \leq d + (k-1)\delta \leq k\delta$.

Now let T be an arbitrary inner primal triangle. Let h and w be its height and width in $\mathcal{T}_{t'}$ and h' and w' its height and width in $\mathcal{T}'$. Then $|h - h'| \leq \delta$ and $|w - w'| \leq 2n\delta$. With a short calculation we then get that the difference of the aspect ratios of T in $\mathcal{T}_{t'}$ and $\mathcal{T}'$ is $|\frac{h}{w} - \frac{h'}{w'}| \leq \frac{\varepsilon'}{3}$. Thus the aspect ratio vector r' of $\mathcal{T}'$ fulfills $\|s_{t'} - r'\|_\infty \leq \frac{\varepsilon'}{3}$.

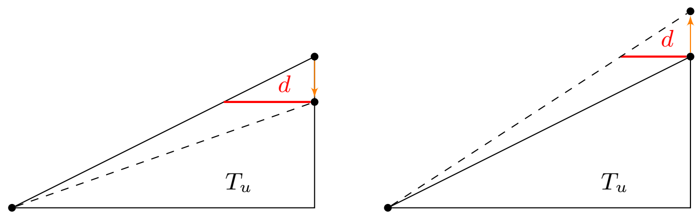


Fig. 12: Change of the x-coordinate of the right segment of T_v (case 2)

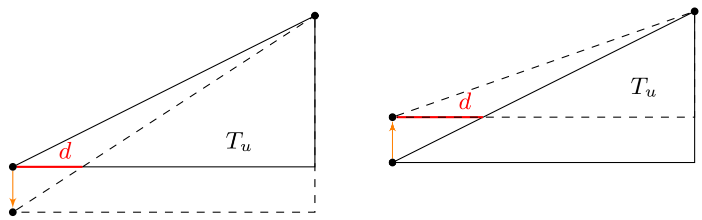


Fig. 13: Change of the x-coordinate of the right segment of T_v (case 3)

It remains to show how to change $\mathcal{T}'$ to a nondegenerate right triangle contact representation $\mathcal{T}''$ with a neighboring Schnyder Wood of S. If the boundary cycle C of f is oriented clockwise in S, we can do this by the reverse above construction for the case that C is oriented counterclockwise, and vice versa. Then the Schnyder Wood S' of $\mathcal{T}''$ is the neighboring Schnyder Wood of S such that the difference cycle of S and S' is C, and the aspect ratio vector r of $\mathcal{T}''$ fulfills $\|r' - r\|_\infty \leq \frac{\varepsilon'}{3}$. Therefore we finally get

$$\|s_{t_0} - r\|_\infty \leq \|s_{t_0} - s_{t'}\|_\infty + \|s_{t'} - r'\|_\infty + \|r' - r\|_\infty \leq \varepsilon' \ .$$

□

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