Pentagon Contact Representations*

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Abstract

Representations of planar triangulations as contact graphs of geometric shapes have received quite some attention in recent years. The most prominent example is Koebe's 'kissing coins theorem', but representations with internally disjoint homothetic triangles or squares have also been studied. In this paper we investigate representations of planar triangulations as contact graphs of a set of internally disjoint homothetic pentagons. Surprisingly, such a representation exists for every triangulation whose outer face is a 5-gon. We relate these representations to *five color forests*. These combinatorial structures resemble Schnyder woods and transversal structures. In particular there is a bijection to certain α-orientations and consequently a lattice structure on the set of five color forests of a given graph. This distributive lattice plays a role in a heuristic that is supposed to compute a contact representation with pentagons for a given graph.

Keywords: Contact representation, pentagon, Schnyder wood, α-orientation.

1 Introduction

A *pentagon contact system* $\mathcal{S}$ is a finite system of convex pentagons in the plane such that any two pentagons intersect in at most one point. If all pentagons

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of $\mathcal{S}$ are regular pentagons with a horizontal segment at the bottom, we call $\mathcal{S}$ a *regular pentagon contact representation*. Note that in this case any two pentagons of $\mathcal{S}$ are homothetic. The contact system is *non-degenerate* if every contact involves exactly one corner of a pentagon. The *contact graph* $G^*(\mathcal{S})$ of $\mathcal{S}$ is the planar graph that has a vertex for every pentagon and an edge for every contact of two pentagons in $\mathcal{S}$. For a given plane graph G and a pentagon contact system $\mathcal{S}$ with $G^*(\mathcal{S}) = G$ we say that $\mathcal{S}$ is a *pentagon contact representation* of G.

We only consider the case that G is an *inner triangulation of a 5-gon*, i.e., the outer face of G is a 5-gon with vertices $a_1, \ldots, a_5$ in clockwise order and all inner faces are triangles. Our interest lies in regular pentagon contact representations of G with the additional property that $a_1, \ldots, a_5$ are represented by touching line segments $s_1, \ldots, s_5$ with internal angles equal to $(3/5)\pi$. Segment s_1 is at the top and horizontal, see Fig. 1.

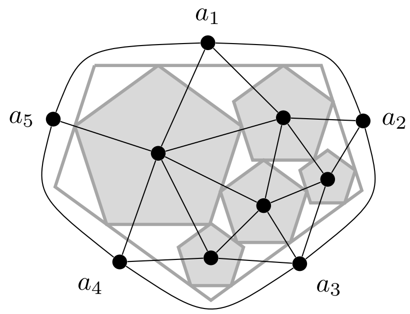

Fig. 1: A regular pentagon contact representation of the graph shown in black.

Triangle contact representations have been introduced by De Fraysseix et al. [3]. They observed that Schnyder woods can be considered as combinatorial encodings of triangle contact representations of triangulations and that any Schnyder wood can be used to construct a corresponding triangle contact system. As an application of the *Monster Packing Theorem*, a strong result of Schramm, it was shown by Gonçalves et al. [7] that 4-connected triangulations admit contact representations with homothetic triangles. A more combinatorial approach to this result which aims at computing the representation as the solution of a system of linear equations which are based on a Schnyder wood was described by Felsner [5]. On the basis of this approach Schrezenmaier [9] reproved the existence of homothetic triangle contact representations.

Representations of graphs using squares or, more precisely, graphs as a tool to model packings of squares already appear in classical work of Brooks et al. [1] from 1940. Schramm [8] proved that every 5-connected inner triangulation of a 4-gon admits a square contact representation. Again there is a combinatorial approach to this result which aims at computing the representation as the solution of a system of linear equations, see Felsner [6]. In this context *transversal structures* play the role of Schnyder woods. As in the case of homothetic triangles this approach is based on an iterative procedure, however, a proof that the iteration terminates is still missing. On the basis of the approach Schrezenmaier [9] reproved Schramm's Squaring Theorem.

In this paper we study representations of planar triangulations as contact graphs of a set of internally disjoint homothetic pentagons. From Schramm's *Monster Packing Theorem* it easily follows that such a representation exists for every triangulation whose outer face is a 5-gon. We relate such representations to *five color forests*. The main part of the paper is reserved to show in what extent these combinatorial structures resemble Schnyder woods and transversal structures. At the end of the paper we propose a heuristic for computing homothetic pentagon representations on the basis of systems of equations and local changes in the corresponding five color forests. The idea of looking for pentagon contact representations and a substantial part of the work originate in the bachelor's thesis of the third author [10].

2 Five Color Forests and α-orientations

In this section G will always be a triangulation with outer face $a_1, \ldots, a_5$ in clockwise order. The set $1, \ldots, 5$ of colors is to be understood as representatives modulo 5, e.g., -1 and 4 denote the same color.

Definition 2.1 *A* five color forest *of G is an orientation and coloring of the inner edges of G in the colors $1, \ldots, 5$ with the following local properties:*

(F1) All edges incident to a_i are oriented towards a_i and colored in color i (see Fig. 2).

(F2) For every inner vertex v the incoming edges build five (possibly empty) blocks B_i of edges of color i with clockwise order $B_1, \ldots, B_5$. Moreover, v has at most one outgoing edge of color i and this edge has to be located between the blocks B_{i+2} and B_{i-2} (see Fig. 2).

(F3) For every inner vertex the block B_i is nonempty or one of the outgoing edges of colors $i-2$ and $i+2$ exists.

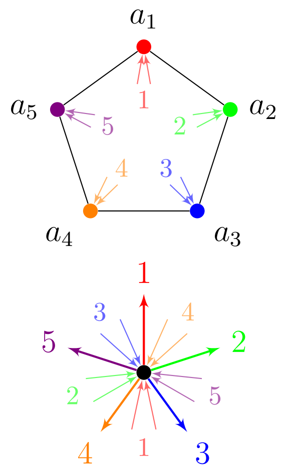

Fig. 2: Properties F1 and F2.

Theorem 2.2 *A regular pentagon contact representation induces a five color forest on its contact graph.*

Ideas of the proof. Let $\mathcal{S}$ be a non-degenerate regular pentagon contact representation of $G = G^*(\mathcal{S})$. Color the corners of pentagons in clockwise order, starting with color 1 at the lowest corner. An inner edge e of G corresponds to the contact of a corner of a pentagon A and an edge of a pentagon B in $\mathcal{S}$. Orient e as $A \to B$ and color it in the color of the corner of A involved in the contact. This yields a five color forest. Property (F3) is fulfilled (e.g., for $i = 1$) because G is a triangulation and therefore for any pentagon A of $\mathcal{S}$ at least one neighboring pentagon has to intersect the area below the horizontal segment of A. Figure 3 shows an example.

In the case that $\mathcal{S}$ is degenerate, each contact of two pentagon corners can be interpreted in two ways as a corner-edge contact with infinitesimal distance to the other corner. We choose one of these interpretations and proceed as before. Hence, the five color forest induced by a degenerate pentagon contact representation is not unique. □

Definition 2.3 *Let H be an undirected graph and $\alpha : V(H) \to \mathbb{N}$. An orientation H' of H is called an* α-orientation *if* $\text{outdeg}(v) = \alpha(v)$ *for all vertices $v \in V(H')$.*

In a five color forest, every inner vertex has outdegree at most 5, but not exactly 5. The following lemma allows to handle this issue.

Lemma 2.4 *Let G be endowed with a five color forest and let f be a face of G that does not contain an outer edge. Then in exactly one of the three interior angles of f a color is skipped.*

Next we define an extension graph of G and a function α such that every five color forest of G can be extended to an α-orientation of this extension.

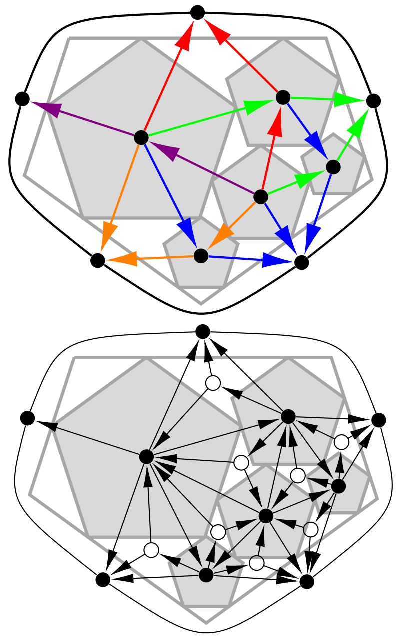

Fig. 3: The five color forest and the α-orientation.

Definition 2.5 *The* stack extension $G^\star$ *of G contains an extra vertex in every inner face that does not contain an outer edge. These new vertices are connected to all three vertices of the respective face. We call the new vertices* stack vertices *and the vertices of G* normal vertices.

From now on, if we talk about α-orientations, we always mean α-orientations of $G^\star$ with $\alpha(v) = 2$ if v is a stack vertex, $\alpha(v) = 5$ if v is an inner normal vertex, and $\alpha(v) = 0$ if v is an outer normal vertex. The outer edges remain undirected. A five color forest on G induces an α-orientation of $G^\star$ in a canonical way by keeping the orientation of the edges of G and defining the missing edge of Lemma 2.4 as the unique incoming edge of every stack vertex.

Observation 2.6 *The coloring of the inner edges of a five color forest can be extended to a coloring of the inner edges of the induced α-orientation that fulfills the properties of a five color forest at all normal vertices.*

To show that the canonical mapping from five color forests to α-orientations is a bijection we have to reconstruct the colors of the inner edges of G when given an α-orientation. The idea is to start with an inner edge e of G and follow a properly defined path that at some point reaches one of the five outer vertices. Then the color of this outer vertex will be the color of e. In the definition of these paths we aim at continuing with the outgoing edge on the opposite side of a vertex. If we run into a stack vertex, there is no unique opposite edge. Therefore, the path of e is not unique, but we can associate a unique outer vertex with e. This approach is similar to the proof of the bijection between Schnyder Woods and 3-orientations in [2].

Now we are able to prove the main result of this subsection.

Theorem 2.7 *The canonical mapping from five color forests to α-orientations is a bijection.*

Ideas of the proof. Let $\mathcal{F}$ be the set of five color forests of G and $\mathcal{A}$ the set of α-orientations of $G^\star$. Further let $\chi : \mathcal{F} \to \mathcal{A}$ be the canonical map and $\psi : \mathcal{A} \to \mathcal{F}$ the map that keeps the orientation of the edges as in the α-orientation and colors every edge as sketched above. Then ψ is well-defined and the inverse function of χ. □

It has been shown in [4] that the set of all α-orientations of a planar graph carries the structure of a distributive lattice. By applying this theory, we can derive the following result about five color forests.

Theorem 2.8 *The set of all five color forests of G carries the structure of a distributive lattice. In this lattice a five color forest F_1 covers a five color forest F_2 if and only if the α-orientation corresponding to F_1 can be obtained from the α-orientation corresponding to F_2 by the reorientation of a counterclockwise oriented facial cycle.*

3 A heuristic

In this section we will propose a heuristic to compute a regular pentagon contact representation of a given graph G.

If we know a five color forest of G that is induced by a regular pentagon contact representation, the following system of linear equations allows us to compute this representation: Every inner vertex v of G gets a variable x_v representing the edge length of the corresponding pentagon and every inner face f gets four variables $x_f^{(1)}, \ldots, x_f^{(4)}$ standing for the four edge lengths of the corresponding quadrilateral in clockwise order where the concave corner is located between the edges corresponding to $x_f^{(1)}$ and $x_f^{(2)}$ (see Fig. 4). For the five inner faces which are incident to two outer vertices of G we simply set $x_f^{(1)} = 0$ since these faces are represented by triangles, not by quadrilaterals. Now every inner vertex v naturally induces five equations, namely that x_v is equal to the sum of the lengths of the face edges building the five edges of the pentagon corresponding to v (this information comes from the five color forest). For geometric reasons the three convex corners of the quadrilaterals corresponding to the faces of G are always exactly $(1/5)\pi$. This implies for every inner face f the two equations

$$x_f^{(3)} = x_f^{(1)} + \phi x_f^{(2)} \ , \ x_f^{(4)} = \phi x_f^{(1)} + x_f^{(2)}$$

where ϕ denotes the golden ratio. Finally, we add one more equation to our system stating that the sum of the lengths of the face edges building the line segment corresponding to the outer vertex a_1 of G is exactly 1.

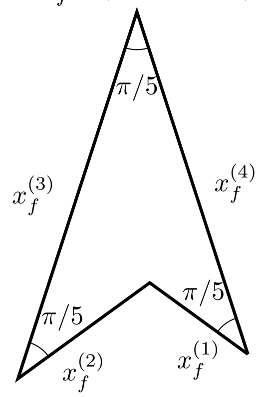

Fig. 4: Variables of an inner face f.

Theorem 3.1 *The above system of linear equations is uniquely solvable for every five color forest. If the solution is nonnegative, it corresponds to a regular pentagon contact representation of G.*

The basic idea of our heuristic is to start with an arbitrary five color forest of G and to solve the system of linear equations. If the solution is nonnegative, we can construct the regular pentagon contact representation from the edge lengths given by the solution and are done. If the solution has negative entries, we would like to change the five color forest and proceed with the new one. To modify the five color forest we identify the edges of $G^\star$ which separate negative and nonnegative variables (we call these edges *sign-separating edges*). It turns out that the sign-separating edges in the orientation of the α-orientation form an Eulerian orientation and therefore can be reversed to obtain a new α-orientation. To prove this we assign to each sign-separating edge e a predecessor which is another sign-separating edge whose

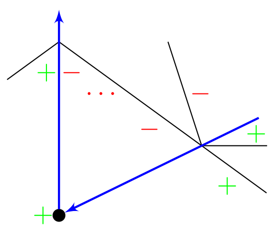
Fig. 5: The predecessor of a sign-separating edge

head is the tail of e (this assignment is injective and thus bijective). The main idea of this construction is that if a vertex-variable corresponding to a pentagon A is non-negative then not all edge-variables of a segment of A can be negative. Therefore, if at a corner of A a negative and a non-negative edge meet, the segment of the negative edge has to switch positive at some point. At this point we can find the predecessor. Figure 5 shows the most simple case of this construction.

We can not prove that iterating this modifications yields any kind of global progress. Therefore, the iteration might continue forever. However, similar heuristics for the computation of homothetic square or triangle contact representations have been subject to extensive experiments. They have always been successful. We therefore conjecture that the proposed heuristic for computing regular pentagon contact representations always terminates with a solution and only needs a polynomial number of iterations to get there.

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