Equiangular polygon contact representations*

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Abstract. Planar graphs are known to have contact representations of various types. The most prominent example is Koebe's 'kissing coins theorem'. Its rediscovery by Thurston lead to effective versions of the Riemann Mapping Theorem and motivated Schramm's Monster Packing Theorem. Monster Packing implies the existence of contact representations of planar triangulations where each vertex v is represented by a homothetic copy of some smooth strictly-convex prototype P_v.

With this work we aim at computable approximations of Schramm representations. For fixed K approximate P_v by an equiangular K-gon Q_v with horizontal basis. From Schramm's work it follows that the given triangulation also has a contact representation with homothetic copies of these K-gons. Our approach starts by guessing a *K-contact-structure*, i.e., the combinatorial structure of a contact representation. From the combinatorial data, we build a system of linear equations whose variables correspond to lengths of boundary segments of the K-gons. If the system has a non-negative solution, this yields the intended contact representation. If the solution of the system contains negative variables, these can be used as sign-posts indicating how to change the K-contact-structure for another try.

In the case $K = 3$ the K-contact-structures are Schnyder woods, in the case $K = 4$ they are transversal structures. As in these cases, for $K \geq 5$ the K-contact-structures of a fixed graph are in bijection to certain integral flows, and can be viewed as elements of a distributive lattice.

The procedure has been implemented, it computes the solution with few iterations. The experiments involved graphs with up to one hundred vertices.

1 Introduction

Representations of graphs by contacts of geometric objects are actively studied in graph theory and geometry. An early result in this direction is Koebe's Circle Packing Theorem from 1936. It states that every planar graph can be represented as the contact system of a set of interiourly disjoint disks. Koebe

* A full version of the paper is available at http://page.math.tu-berlin.de/~felsner/Paper/kgons.pdf

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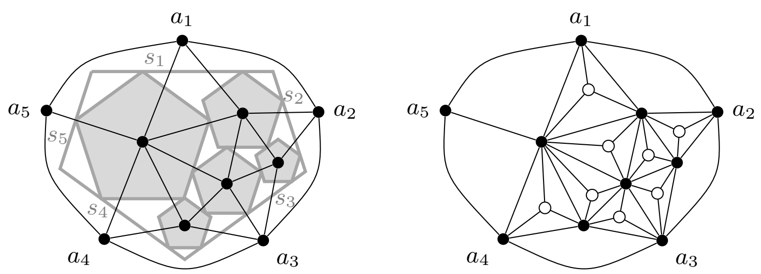

Fig. 1. Left: An equiangular pentagon contact representation of the graph G shown in black where each inner vertex is represented by a regular pentagon. Right: The stack extension $G^\star$ of G.

arrived at this result in the context of conformal mapping of 'contact domains'. Unaware of Koebe's work, Thurston reproved the Circle Packing Theorem and connected it to the Riemann Mapping Theorem. This line of research resulted in discretizations of conformal mappings and has strong impact in the area of discrete differential geometry. We refer to [19] and [1] for further details on those connections.

A very strong generalization of Koebe's theorem is Schramm's Convex Packing Theorem from 1990 [14]. The theorem states that if each vertex v of a planar triangulation G has a prescribed convex prototype P_v, then there is a contact representation of G where each vertex is represented by a (possibly degenerate) homothet of its prototype. When the prototypes have a smooth boundary there are no degeneracies. With this work we aim at efficiently computable approximations of Schramm representations. The idea is to approximate the prototypes P_v with simpler shapes; we use equiangular K-gons. Clearly, a sequence of approximating contact representations with K-gons, one for each positive integer K and each of them confined to the unit square, will contain a subsequence converging to a representation with the prototypes P_v.

Contact representations of graphs with polygons have also been studied widely. Triangle contact representations have been investigated by De Fraysseix et al. [5]. They observed that Schnyder woods can be considered as combinatorial encodings of triangle contact representations of triangulations and that any Schnyder wood can be used to construct a corresponding triangle contact system. Gonçalves et al. [11] observed that Schramm's Convex Packing Theorem can be used to prove the existence of contact representations with homothetic triangles for all 4-connected triangulations. A more combinatorial approach to this result, which aims at computing the representation as the solution of a system of linear equations, which are based on a Schnyder wood, was described by Felsner [7]. On the basis of this approach, Schrezenmaier [17] reproved the existence of homothetic triangle contact representations.

Representations of graphs with side contacts of rectangles have applications in architecture and VLSI design. For links into the extensive literature we recommend [3] and [8]. Representations of graphs using squares or, more precisely, graphs as a tool to model packings of squares already appear in classical work of

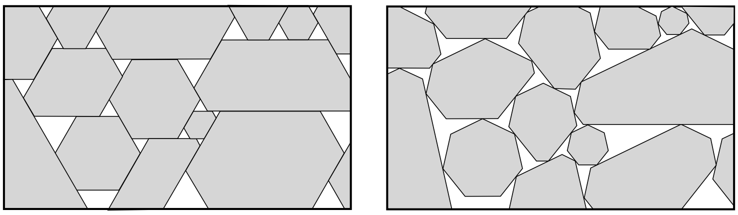

Fig. 2. Parts of equiangular 6-gon and 7-gon contact representations of the same graph.

Brooks et al. [2] from 1940. Schramm [15] proved that every 5-connected inner triangulation of a 4-cycle admits a square contact representation. Again there is a combinatorial approach to this result which aims at computing the representation as the solution of a system of linear equations, see Felsner [8]. In this context *transversal structures* play the role of Schnyder woods. As in the case of homothetic triangles, this approach is based on an iterative procedure, however, a proof that the iteration terminates is still missing. On the basis of the approach, Schrezenmaier [16] reproved Schramm's Squaring Theorem.

Before stating our results, we introduce some precise terminology. A *K-gon contact system* $\mathcal{S}$ is a finite system of convex K-gons in the plane such that the interiors of any two K-gons are disjoint. If all K-gons of $\mathcal{S}$ are equiangular K-gons (i.e., all interior angles are $\frac{K-2}{K}\pi$) with a horizontal segment at the bottom, we call $\mathcal{S}$ an *equiangular K-gon contact representation.* The contact system has an *exceptional touching* if there is a point where two corners of K-gons meet. The *contact graph* $\mathcal{G}(\mathcal{S})$ of $\mathcal{S}$ is the graph that has a vertex for every K-gon and an edge for every contact of two K-gons in $\mathcal{S}$. Note that $\mathcal{G}(\mathcal{S})$ inherits a crossing-free embedding from $\mathcal{S}$. For a given plane graph G and a K-gon contact system $\mathcal{S}$ with $\mathcal{G}(\mathcal{S}) = G$ we say that $\mathcal{S}$ is a *K-gon contact representation* of G.

We will only consider the case that G is an *inner triangulation of a K-cycle*, i.e., the outer face of G is a K-cycle with vertices $a_1, \ldots, a_K$ in clockwise order, all inner faces are triangles, there are no loops or multiple edges, and there are no additional edges between the outer vertices. Our interest lies in regular K-gon contact representations of G with the additional property that $a_1, \ldots, a_K$ are represented by line segments $s_1, \ldots, s_K$ which together form an equiangular K-gon. The line segment s_1 is always horizontal and at the top, and $s_1, \ldots, s_K$ is the clockwise order of the segments of the K-gon. Figure 1 (left) shows an example for $K = 5$. Figure 2 shows contact systems of 6-gons and 7-gons, respectively.

Let G be an inner triangulation of a K-cycle and for each inner vertex v of G let P_v be a prescribed equiangular K-gon. From Schramm's Convex Packing Theorem it follows that G has a representation as contact graph of homothets of the prototypes (see Section 2). The representation is non-degenerate whenever $K \geq 5$ and odd, or $K \geq 8$ and even. For $K = 3$ and $K = 6$ the graph needs to be 4-connected to guarantee a non-degenerate representation. This is because the three K-gons corresponding to a triangle in G can touch in a single point such that there is no space left for the K-gons of vertices in the interior of this triangle.

We propose a new method for computing equiangular K-gon contact representations. The idea is to guess the combinatorial structure of the representation of G, i.e., for each edge uv of G guess whether the contact involves a corner of P_u or a corner of P_v and also guess which corner of the respective prototype is involved. The guess is encoded in a *K-contact-structure*. The K-contact-structure leads to a system of linear equations whose variables correspond to lengths of boundary segments of the K-gons. The system is non-singular. If it has a non-negative solution, the values of the variables determine the geometry of a K-gon contact representation. If the solution of the system contains negative values, then it is possible to locally modify the K-contact-structure in the neighborhood of negative variables. The modified K-contact-structure corresponds to a new system of equations which has a new solution. This yields an iterative procedure which *hopefully* stops with a positive solution, i.e., with a K-gon contact representation.

We could not prove that the above iterative procedure stops. However the algorithm has been implemented and was used for extensive experiments (Section 7). These have always been successful. Similar algorithms for the computation of contact representations by homothetic triangles or squares have been described by Felsner [7, 8]. These have also been implemented and successfully tested, c.f. Rucker [13] and Piccetti [12], respectively. We therefore conjecture that the proposed algorithm for computing equiangular K-gon contact representations always terminates with a solution.

In Section 3 we introduce K-contact-structures of the graph G. These are certain weighted orientations of a supergraph of G. In Section 4 we enhance K-contact-structures with a K-coloring of the edges. The color classes are directed forests that resemble the trees of a Schnyder wood. In Section 5 we show that there is a distributive lattice on the set of K-contact-structures of a fixed graph G and describe the combinatorial change in K-contact-structures that form a cover pair. In Section 6 we discuss the system of linear equations and prove that it is non-singular. Section 7 describes the iteration which is proposed as a heuristic for computing equiangular K-gon contact representations.

In this paper we focus on odd $K \geq 5$. The case $K = 3$ is well-studied and the case $K \geq 6$ and even will be added in a later version of this paper. The case $K = 5$ was first studied in the bachelor thesis of Steiner [18] (a coauthor in this paper) and further elaborated by the present team of authors [10].

2 The existence of equiangular K-gon contact representations

In this section let G be an inner triangulation of a K-cycle and let V_{inner} be the set of inner vertices of G. Further, for each $v \in V_{\text{inner}}$, let P_v be an equiangular K-gon with a horizontal segment at the bottom. We call P_v the *prototype* of v. A *homothetic* copy of a prototype P_v is a set in the plane that can be obtained from P_v by scaling and translation.

Theorem 1. *For odd $K \geq 5$ there is an equiangular K-gon contact representation of G in which each $v \in V_{\text{inner}}$ is represented by a homothetic copy of P_v.*

This theorem is an immediate consequence of the Convex Packing Theorem by Schramm [14] which guarantees a contact representation of G with homothets of the given prototypes if we also allow the inner vertices to be represented by a single point, i.e., a homothetic copy of the prototype with scaling factor 0. The interesting point is that for odd $K \geq 5$ this cannot happen because the interior angles of the equiangular K-gons are too large (combined with the fixed alignment of the K-gons) to allow more than two equiangular K-gons to meet at a given point. Similar proofs have been given for the case $K = 3$ in [11] and $K = 5$ in [18] and [10].

3 The combinatorial structure of equiangular polygon contact representations

For the entire section let G be an inner triangulation of a K-cycle, $K \geq 3$ odd. We call an inner face of G a *strictly inner face* if it is only incident to inner edges. We denote the set of inner edges of a planar graph H by $E_{\text{inner}}(H)$. For the directed graphs used later in this section we denote the sets of incoming and outgoing edges of a vertex v by $E_{\text{in}}(v)$ and $E_{\text{out}}(v)$, respectively.

Definition 1. *The* stack extension $G^\star$ *of G is the extension of G that contains an extra vertex in every strictly inner face. These new vertices are connected to all three vertices of the respective face. We call the new vertices* stack vertices *and the vertices of G* normal vertices*. Analogously, we call the new edges* stack edges *and the edges of G* normal edges*. See Fig. 1 (right) for an example.*

Definition 2. *A* K-contact-structure *on G is an orientation and weighting $w : E_{\text{inner}}(G^\star) \to \mathbb{N}$ of the inner edges of $G^\star$ such that*

(P1) $w(e) = 1$ for each normal edge e,
(P2) each stack edge is oriented towards its incident stack vertex,
(P3) the out-flow of each normal vertex u is K, i.e., $\sum_{e \in E_{\text{out}}(u)} w(e) = K$,
(P4) the in-flow of each stack vertex v is $\frac{K-3}{2}$, i.e., $\sum_{e \in E_{\text{in}}(v)} w(e) = \frac{K-3}{2}$.

Definition 3. *Let $\mathcal{A}$ be a K-contact-structure on G. Then we can associate with $\mathcal{A}$ a modified version of $G^\star$ where each inner edge e is replaced by $w(e)$ parallel edges (if $w(e) = 0$, the edge e is deleted) and all edges are oriented as in $\mathcal{A}$. We denote this graph by $G^\star_+(\mathcal{A})$.*

The following theorem shows the key correspondence between K-contact-structures and equiangular K-gon contact representations.

Theorem 2. *Let $\mathcal{S}$ be an equiangular K-gon contact representation of the graph $G = \mathcal{G}(\mathcal{S})$. Then $\mathcal{S}$ induces a K-contact-structure on G (see Fig. 3 for an illustration).*

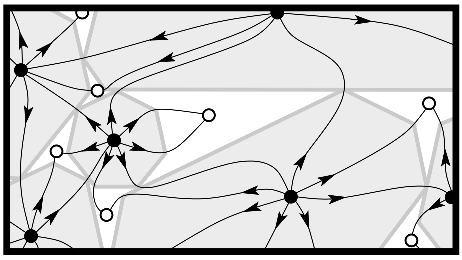

Fig. 3. A contact representation of equiangular 7-gons and the graph $G^\star_+(\mathcal{A})$ for its induced 7-contact-structure $\mathcal{A}$.

First we consider the case that $\mathcal{S}$ has no exceptional touchings. Then the construction of the induced K-contact-structure of $\mathcal{S}$ is as follows: Let e be an inner normal edge of $G^\star$. Then e corresponds to the contact of a corner of a K-gon A and a segment of a K-gon B in $\mathcal{S}$. We orient the edge e from the vertex corresponding to A to the vertex corresponding to B and set $w(e) = 1$. Now let $e = uv$ be a stack edge with normal vertex u and stack vertex v. Then u corresponds to a K-gon A of $\mathcal{S}$ and v to an area F in $\mathcal{S}$ which is enclosed by A and two more K-gons or outer segments s_i. Note that F is a pseudotriangle, i.e., a polygon with exactly three convex corners and arbitrarily many concave corners. We define $w(e)$ to be the number of concave corners of F which are also corners of A, and orient e from u to v.

Properties (P1) and (P2) are fulfilled directly by construction. Property (P3) corresponds to the fact that each K-gon has exactly K corners, and property (P4) corresponds to the fact that each pseudotriangle has exactly $\frac{K-3}{2}$ concave corners.

In the case that $\mathcal{S}$ has exceptional touchings, each exceptional touching of two K-gon corners can be interpreted in two ways as a corner-segment contact with infinitesimal distance to the other corner. We choose one of these interpretations and proceed as before. Hence, the K-contact-structure induced by an equiangular K-gon contact representation with exceptional touchings is not unique.

Theorem 3. *Let G be an inner triangulation of a K-cycle. Then there exists a K-contact-structure on G.*

Theorem 3 immediately follows from Theorem 1 and Theorem 2. Since we aim for a theory independent from the Monster Packing Theorem by Schramm, we give another elementary proof of Theorem 3. The idea of the proof is the following: We replace each stack edge of $G^\star$ by $\frac{K-3}{2}$ parallel edges. Then we show that there exists an orientation of this graph such that each normal vertex has out-degree K and each stack vertex has in-degree $\frac{K-3}{2}$. Such orientations with prescribed vertex degrees have been studied in [6] under the name of *α-orientations*. There are sufficient conditions related to Hall's matching criterion for the existence of α-orientations. In our case the conditions are fulfilled. The existence of the appropriate orientation implies the existence of a K-contact-structure.

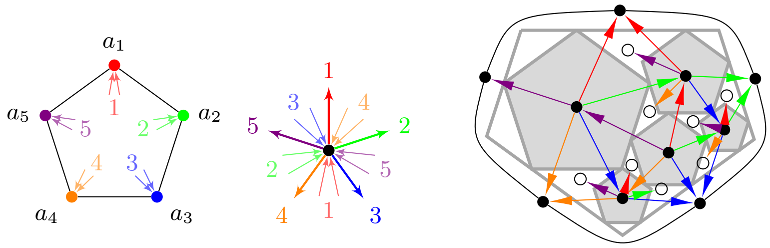

Fig. 4. Left: The local conditions of a K-proper coloring in the case $K = 5$. Right: An example for the K-proper coloring of an induced K-contact-structure.

4 Coloring K-contact-structures

In this section let G be an inner triangulation of a K-cycle, let $\mathcal{A}$ be a K-contact-structure on G and let $G_+^\star := G_+^\star(\mathcal{A})$. In the following, the set of colors $1, \ldots, K$ is to be understood as representatives modulo K, i.e., colors c and $c + zK$ are the same for any $z \in \mathbb{Z}$.

Definition 4. *A* K-proper coloring *of* $G_+^\star$ *is a coloring of the inner edges of* $G_+^\star$ *in the colors* $1, \ldots, K$ *such that (see Fig. 4 (left) for an illustration)*

(C1) for $i = 1, \ldots, K$ *all edges incident to the outer vertex* a_i *have color* i*,*
(C2) each normal vertex has exactly one outgoing edge in each color and the clockwise order of the colors is $1, \ldots, K$*,*
(C3) incoming edges of a normal vertex, which are located between the outgoing edges of colors c *and* $c + 1$*, have color* $c - \frac{K-1}{2}$*.*

An equiangular K-gon contact representation $\mathcal{S}$ induces a K-contact-structure together with a K-proper coloring. To see this, recall the construction below Theorem 2. Each inner edge of $G_+^\star$ corresponds to a corner of a K-gon of $\mathcal{S}$. We color the corners of each K-gon of $\mathcal{S}$ in the colors $1, \ldots, K$ in clockwise order, starting with color 1 at the corner at the top. Then each inner edge of $G_+^\star$ gets the color of the corner it corresponds to. Figure 4 (right) shows an example.

The following theorem shows that this coloring is a property of the K-contact-structure itself and independent of an inducing contact representation.

Theorem 4. *The graph* $G_+^\star$ *has a unique* K*-proper coloring.*

The idea of the construction of the colors is as follows: We start with an inner edge e of $G_+^\star$ and follow a properly defined path that at some point reaches one of the outer vertices. Then the color of this outer vertex will be the color of e. This approach is similar to the proof of the bijection of Schnyder Woods and 3-orientations in [4]. In the definition of these paths, we aim at continuing with the outgoing edge on the opposite side of a vertex. This is motivated by the following geometric idea: If we are already given an equiangular K-gon contact representation, such paths keep a constant slope and therefore run into an outer segment with corresponding slope. If we run into a stack vertex, there is no unique opposite edge. Therefore the path of e is not unique, but we can associate a unique outer vertex with e by showing that all properly defined paths starting with e end in the same outer vertex.

5 The distributive lattice of K-contact-structures

Let G be an inner triangulation of a K-cycle. The following definitions give us a formalism how to change a K-contact-structure of G to obtain a new one.

Definition 5. *Let $\mathcal{A}$ be a K-contact-structure of G. We call a multiset E of oriented edges of $G^\star$* flippable *in $\mathcal{A}$ if i) $\mathrm{indeg}_E(v) = \mathrm{outdeg}_E(v)$ for each vertex v; ii) each normal edge is contained at most once in E and only in the orientation of $\mathcal{A}$; iii) each stack edge $e = uv$ with stack vertex v is contained at most $w_{\mathcal{A}}(e)$ times in E in the orientation from u to v (no restriction for the other direction).*

Definition 6. *Let $\mathcal{A}$ be a K-contact-structure of G and let E be a flippable set of edges in $\mathcal{A}$. Then we can perform a* flip *on $\mathcal{A}$ and obtain a new K-contact-structure $\mathcal{A}'$ by changing the orientation of all normal edges in E, and by setting $w_{\mathcal{A}'}(e) := w_{\mathcal{A}}(e) - a + b$ for each stack edge $e = uv$ with normal vertex u and stack vertex v if e is contained a times in E oriented from u to v and b times oriented from v to u.*

It can easily be seen that a flip indeed yields a new K-contact-structure. We can even reach every K-contact-structure $\mathcal{A}'$ from $\mathcal{A}$ by flipping a suitable flippable set of edges.

These flipping operations already show the close relation between K-contact-structures and integral flows on $G^\star$. We now want to formalize this relation and thereby obtain the structure of a distributive lattice on the set of K-contact-structures of G. In particular, K-contact-structures can be equivalently modeled as flows $f : E_{\mathrm{inner}}(\overrightarrow{G^\star}) \to \mathbb{Z}$ on a fixed orientation $\overrightarrow{G^\star}$ of $G^\star$ where each stack edge is oriented towards the incident stack vertex and each normal edge obtains an arbitrary fixed orientation. In such a flow the *excess* of a vertex v is defined as $\omega(v) := \sum_{e \in E_{\mathrm{in}}(v)} f(e) - \sum_{e \in E_{\mathrm{out}}(v)} f(e)$.

Definition 7. *A flow $f : E_{\mathrm{inner}}(\overrightarrow{G^\star}) \to \mathbb{Z}$ is called a* K-contact-flow *if*

(i) $f(e) \in \{0, 1\}$ *for each normal edge e;*
(ii) $f(e') \in \{0, \ldots, \frac{K-3}{2}\}$ *for each stack edge e';*
(iii) $\omega(u) = \mathrm{indeg}(u) - K$ *for each normal vertex u;*
(iv) $\omega(v) = \frac{K-3}{2}$ *for each stack vertex v.*

For each normal edge e we set $c_l(e) := 0$ and $c_u(e) := 1$. For each stack edge e' we set $c_l(e') := 0$ and $c_u(e') := \frac{K-3}{2}$. Then the first two conditions can be formulated as $c_l(e'') \le f(e'') \le c_u(e'')$ for each edge e''. The set of integral flows $\mathcal{F}(H, \omega, c_l, c_u)$ of a directed planar graph H fulfilling such constraints (bounds c_l, c_u on the flow values and prescribed excesses ω) has been studied in [9].

The following describes a bijection between the set of K-contact-structures and the set of K-contact-flows of G. Let $\mathcal{A}$ be a K-contact-structure on G. If a normal edge e has the same orientation in $\overrightarrow{G^\star}$ and in $\mathcal{A}$, we set $f(e) = 1$, otherwise $f(e) = 0$. For a stack edge e' we set $f(e') = w_{\mathcal{A}}(e')$.

It has been shown in [9] that the set $\mathcal{F}(H, \omega, c_l, c_u)$ carries the structure of a distributive lattice. For a flow $f \in \mathcal{F}(H, \omega, c_l, c_u)$ let the *residual graph*, denoted by H_f, be the following reorientation of H: An edge vw of H is oriented from v to w in H_f if $f(vw) > c_l(vw)$ and from w to v if $f(vw) < c_u(vw)$. Note that in H_f an edge can have no, one, or two orientations. If we decrease the flow f by one on a subgraph H'_f of H_f with the property $\mathrm{indeg}_{H'_f}(v) = \mathrm{outdeg}_{H'_f}(v)$ for each vertex v, we obtain a new flow $f' \in \mathcal{F}(H, \omega, c_l, c_u)$. This operation corresponds to a flip in the K-contact-structure.

Definition 8. *A* chordal path *of a simple cycle C is a directed path consisting of edges inside C (referring to a plane embedding) whose first and last vertex are vertices of C. These two vertices are allowed to coincide.*

A simple cycle C is an essential cycle *if there is a flow f such that C is a directed cycle in H_f and has no chordal path in H_f.*

Theorem 5 ([9]). *The following relation on the set $\mathcal{F}(H, \omega, c_l, c_u)$ of flows of a planar graph H is the cover relation of a distributive lattice: A flow f' covers a flow f if and only if f' can be obtained from f by subtracting one unit of flow on a counterclockwise oriented essential cycle in H_f.*

Now we can apply this to the set of K-contact-flows of G.

Theorem 6. *The set of all K-contact-structures of G carries the structure of a distributive lattice. In this lattice a K-contact-structure $\mathcal{A}'$ covers a K-contact-structure $\mathcal{A}$ if there is a flippable counterclockwise oriented facial cycle in $G^\star$ such that $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by flipping this cycle.*

6 System of linear equations

Let G be an inner triangulation of a K-cycle and let $\mathcal{A}$ be a K-contact-structure of G. Let $G^\star_+ := G^\star_+(\mathcal{A})$. We will propose a system of linear equations that allows us to compute an equiangular K-gon contact representation of G with induced K-contact-structure $\mathcal{A}$ if such a representation exists. If such a representation does not exist, the solution of the system will have negative variables.

We start by describing how to obtain the skeleton graph of the contact representation. The skeleton graph G_{skel} is the medial graph of $G^\star_+$ without the edges corresponding to the outer face of $G^\star_+$. We color the edges of G_{skel} according to the following rules: If the edge corresponds to an angle of an inner normal vertex and this angle lies between the outgoing edges of colors c and $c+1$ in the K-proper coloring of $G^\star_+$, it gets the color $c - \frac{K-1}{2}$. If the edge corresponds to an angle of the outer normal vertex a_i, it gets the color i. See Fig. 5 (left) for an example.

The colors of the edges of G_{skel} correspond to their required slopes in the following way: Let B be an equiangular K-gon with a horizontal side at the top and its sides colored in the colors $1, \ldots, K$ in clockwise order, starting at the top. Then a crossing-free straight-line drawing of G_{skel} is an equiangular K-gon contact representation of G with induced K-contact-structure $\mathcal{A}$ if and only if each edge e has the same slope as the side of B that has the same color as e.

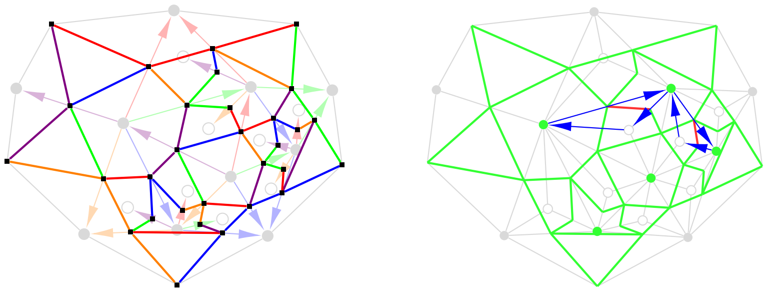

Fig. 5. Left: The skeleton graph corresponding to the K-contact-structure of Fig. 4. Right: The signs of the variables in the solution, green for non-negative and red for negative, and the corresponding sign-separating edges in blue.

The purpose of the system of linear equations is to find edge lengths for the edges of G_{skel} in such a drawing with the additional property that the K-gons are homothets of the given prototypes. Therefore we have a variable x_e for each edge e of G_{skel} representing its length, and a variable x_v for each inner vertex v of G representing the scaling factor of its prototype P_v. We have equations which ensure that the scaling factor x_v of each normal vertex fits together with the edge lengths x_e of the K-gon corresponding to v. For $i = 1, \ldots, K$ let $\ell_i(P_v)$ be the length of the ith segment of P_v, starting with the horizontal segment and then proceeding in clockwise direction. Further let $E_i(v)$ be the edges of color i in G_{skel} corresponding to angles of v. Then the sum of the lengths of the edges in $E_i(v)$ has to be equal to $x_v \ell_i(P_v)$, the scaled segment length of the prototype: $\sum_{e \in E_i(v)} x_e - \ell_i(P_v) x_v = 0$. Further we have 2 equations for each inner face of G ensuring that the edges of G_{skel} corresponding to this face form a closed curve (these are the pseudotriangles). Finally, we add one more equation to our system stating that the lengths of the edges building the line segment corresponding to the outer vertex a_1 of G sum up to 1: $\sum_{e \in E_1(a_1)} x_e = 1$. This equation is the only inhomogeneous equation and will ensure that the solution of the system is unique. We denote the entire system by $A_{\mathcal{A}} \mathbf{x} = \mathbf{e_1}$ where $A_{\mathcal{A}}$ is a matrix depending on the K-contact-structure $\mathcal{A}$ and $\mathbf{e_1} = (1, 0, 0, \ldots, 0)$.

Theorem 7. *The system $A_{\mathcal{A}} \mathbf{x} = \mathbf{e_1}$ has a unique solution. This solution is non-negative if and only if the K-contact-structure $\mathcal{A}$ is induced by an equiangular K-gon contact representation of G with the given prototypes.*

One direction of the latter part of the statement is trivial because the edge lengths of a contact representation yield a non-negative solution of the system. For the other direction we show that we can construct a contact representation from the edge lengths given by a non-negative solution by gluing together the K-gons and pseudotriangles resulting from this solution.

7 A heuristic

In this section we propose a heuristic to compute an equiangular K-gon contact representation of a given triangulation G of a K-cycle. The basic idea of

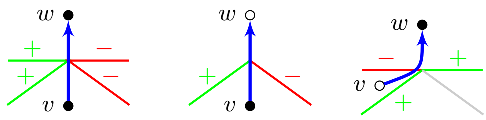

Fig. 6. The sign-separating edges of types (A), (B) and (C).

our heuristic is to start with an arbitrary K-contact-structure $\mathcal{A}$ of G and to solve the system $A_{\mathcal{A}}\mathbf{x} = \mathbf{e_1}$. If the solution is non-negative, we can construct the contact representation from the edge lengths given by the solution and are done. Otherwise, we can use the negative variables of the solution as sign-posts indicating how to change the K-contact-structure for another try. The goal is to find a flippable set of edges in $G^\star$ (see Section 5) that separates the edges of G_{skel} with negative solution value from these with non-negative solution value.

Definition 9. *We call these three types of oriented edges $e = (v, w)$ in $G^\star$* sign-separating *edges (see Fig. 6): (A) v, w are normal vertices, the abstract K-gons of both vertices have a sign-change at the contact, the two involved abstract pseudotriangles do not have a sign-change at the contact; (B) v is a normal vertex, w is a stack vertex, and there is a sign-change at the corner corresponding to e; (C) v is a stack vertex, w is a normal vertex, the abstract pseudotriangle corresponding to v has a sign-change in a convex corner, the abstract K-gon corresponding to w has a sign-change at the same point, but not a corner.*

Let E_{+-} be the set of all sign-separating edges. It might be that there is a normal vertex v and a stack vertex u such that $(u, v), (v, u) \in E_{+-}$. In this case let w be the normal vertex corresponding to the abstract K-gon touching the abstract K-gon of v in the contact point where (u, v) and (v, u) have their assigned sign-changes. Then we change E_{+-} in the following way: We remove (v, u) from E_{+-} and add (v, w) and (w, u) instead. We call this a *repairing step*. It guarantees that a flip of the edges in E_{+-} changes the K-contact-structure.

Theorem 8. *After performing all possible repairing steps, the set E_{+-} is flippable in $\mathcal{A}$ and the corresponding flip leads to a K-contact-structure $\mathcal{A}' \neq \mathcal{A}$.*

We could not prove that iterating to flip the edges in E_{+-} can guarantee any kind of progress. Therefore a proof is still missing that this heuristic always terminates with a solution. However, the heuristic has been subject to extensive experiments. We tested the heuristic with a total of 1000 random graphs with up to 100 inner vertices and up to 23 outer vertices. The heuristic terminated for each graph after few seconds.[3] Therefore we have the following conjecture.

Conjecture 1. The heuristic described in this section terminates with a solution for all K, for every graph G which is an inner triangulation of a K-cycle, and for every K-contact-structure of G to start the heuristic.

[3] Visualizations of some examples can be found at `https://www3.math.tu-berlin.de/diskremath/research/kgon-representations/index.html`.

The experiments also suggest that the number of iterations is polynomial or even linear (sub-linear in the average case). Since the equation system has linear size and systems of linear equations can be solved exactly in polynomial time, this would imply that the heuristic has polynomial running time.

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