9. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 14. June 2017
Due dates: 20.-22. June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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(1) Let G be a graph with a matching M. Color the edges of M blue and all other edges red. A vertex v is *exposed* if all adjacent edges are red, i.e., do not belong to the matching. A path between two vertices is *alternating* if the colors of its edges are alternating in red and blue.
Show that a matching is maximum if and only if no pair of exposed vertices v, w has an alternating path.

(2) Consider two magicians M_1, M_2 in well separated rooms. A volunteer picks five cards from a standard deck (52 cards) and hands them to M_1. M_1 keeps one of the five cards and puts the other four (in specific order) in an envelope. The envelope is brought to M_2 who opens it, has a look at the cards, mumbles, and announces the fifth card.

(a) Explain the existence of a strategy for this trick with the aid of Hall's Theorem.

(*) Find a playable strategy (which you can demonstrate with a colleague).

(3) The following is an incorrect proof of Dilworth's Theorem. Find the mistake:
Induction on $n := |P|$; $n = 1$ is obvious. For the induction step $n \to (n+1)$ let us assume the theorem holds for posets of n elements. By the *width* of P, we mean the size of a maximum antichain. Let $m \in P$ be a maximum of P. Apply the hypothesis to $P \setminus \{m\}$ and gain a decomposition into chains $C_1, \ldots, C_w$ of $P \setminus \{m\}$, with $w = \text{width}(P \setminus \{m\})$. If $w < \text{width}(P)$, then by adding $C_{w+1} = \{m\}$ as an additional chain, we gain a chain decomposition of P. If $w = \text{width}(P)$, the set $\{\max(C_i) \mid i = 1, \ldots, w\} \cup \{m\}$ cannot be an antichain. Therefore, $m \geq \max(C_i)$ for some $i = 1, \ldots, w$. Now $C_i \cup \{m\}$ is a chain, so $C_1, \ldots C_{i-1}, C_i \cup \{m\}, C_{i+1}, \ldots C_w$ is a chain decomposition of P. Thus, in any case, we have a chain decomposition of P with $\text{width}(P)$ chains.

(4) Let $(P, \leq)$ be a poset and $\max((P, \leq)) := \{x \in P \mid x \leq y \Rightarrow y = x\}$ be the set of its maxima. Let $e((P, \leq))$ be the number of linear extensions of $(P, \leq)$. Prove

$$e((P, \leq)) = \sum_{x \in \max(P)} e((P \setminus \{x\}, \leq')),$$

where $\leq'$ is the restriction of $\leq$ to $P \setminus \{x\}$, i.e. $\leq' := \leq \cap (P \setminus \{x\}) \times (P \setminus \{x\}) \subseteq P \times P$.

(5) Choose one of the following exercises:

(i) Consider the poset P_n on the set $\left\{a_1, \ldots, a_{\lceil \frac{n}{2} \rceil}, b_1, \ldots, b_{\lfloor \frac{n}{2} \rfloor}\right\}$ with the cover relations $a_i < a_{i+1}$, $b_i < b_{i+1}$, and $b_i > a_{i-1}$ as well as $a_i > b_{i-2}$ for all i. Determine $e(P_n)$.

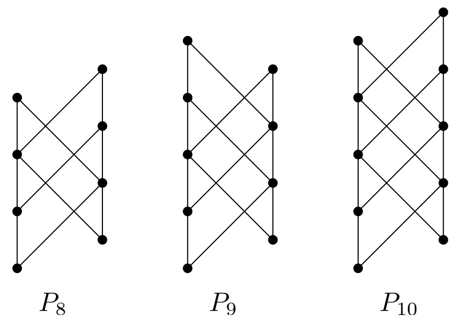

Figure 1: Hasse diagrams of P_8, P_9 and P_{10}

(ii) The Catalan numbers fulfill the recursion $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ with the initial condition $C_0 = 1$. Prove this recursion for $e(Q_n)$ where Q_n is the 'ladder' poset from the lecture for that we showed $e(Q_n) = C_n$.

(iii) Figure 2 shows the posets R_2 to R_5. Let R_n be the poset with $2n+2$ elements in this sequence. Determine $e(R_n)$.

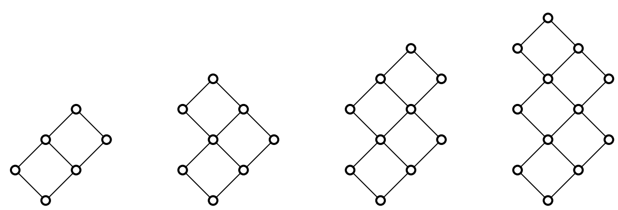

Figure 2: Hasse diagrams of R_2, R_3, R_4 and R_5

(iv) The poset Z_n consists of the elements $x_1, \ldots, x_n$ and the relations $x_{2i-1} < x_{2i}$ und $x_{2i} > x_{2i+1}$ for all i. This poset is called the 'zig-zag' poset, or *fence*. Let $E(n) := e(Z_n)$. Prove that $E(n)$ fulfills the recursion

$$E(n+1) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i-1} E(2i-1)\, E(n-2i+1) \ .$$

The numbers $E(n)$ are called Euler numbers. The first Euler numbers are

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, \ldots$$

The Euler numbers have the nice exponential generating function

$$\sum_{k=0}^{n} \frac{E(n)}{n!} = \sec(x) + \tan(x) \ .$$

More about the Euler numbers can be found at `http://oeis.org/A000111`.