

# Optimal basket liquidation with finite time horizon for CARA investors

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## Abstract

We consider the finite-time optimal basket liquidation problem for a von Neumann-Morgenstern investor with constant absolute risk aversion (CARA). As underlying continuous-time liquidity model, we use a multi-asset extension of the nonlinear price impact model of Almgren (2003). We show that the expected utility of sales revenues, taken over a large class of adapted strategies, is maximized by a unique deterministic strategy, which can be characterized by a Hamilton equation. Mathematically, the problem is equivalent to a control problem with finite-fuel constraint, which we solve by observing that the corresponding value function solves a degenerate Hamilton-Jacobi-Bellman equation with singular initial condition.

## 1 Introduction

A common problem for stock traders is to unwind large block orders of shares. These can comprise a major part of the daily traded volume of shares and create substantial impact on the asset price. The overall costs of such a liquidation can be significantly reduced by splitting the order into smaller orders that are spread over a certain time period. Thus, one possible question is to find optimal allocations for each individual placement such that the *expected* overall liquidity costs are minimized. Problems of this type were analyzed for various market models by Bertsimas and Lo [9], Obizhaeva and Wang [18], and Alfonsi et al. [2, 3], to mention only a few.

Taking the expected liquidity costs as a target function, however, misses the volatility risk that is associated with delaying an order. Almgren and Chriss [5, 6] therefore suggested to replace the minimization of expected costs by a mean-variance optimization for sales revenues and

they solved the corresponding optimization problem in the class of deterministic—or static—strategies; see also Almgren [4]. Almgren and Lorenz [7] allowed for adaptive strategies in mean-variance optimization and found that intertemporal updating can indeed strictly improve mean-variance performance.

The proper economic motivation for mean-variance optimization stems from a second-order approximation of an expected utility functional as explained, e.g., in [13, p. 67]. We therefore propose to study directly the original problem of expected-utility maximization, and we will do this here for exponential, or CARA, utility functions. Apart from being economically appropriate, this approach ensures that optimal strategies are a priori time-consistent, which is important for practical applications. As an underlying market impact model we use a multi-asset extension of the nonlinear impact model proposed by Almgren [4].

Our main result states that for CARA investors there is surprisingly no added utility from allowing for intertemporal updating of strategies, i.e., the expected utility is maximized by a *deterministic* strategy, which can be characterized as the unique solution to a Hamilton ODE. In particular, the mean-variance maximizing strategy obtained in [7] will actually *decrease* the expected value of the exact utility. For practitioners, our result provides an explanation of the wide use of deterministic liquidation strategies. Furthermore, the Hamilton ODE can be used to numerically derive the optimal deterministic liquidation strategy.

Mathematically, the problem can be set up as a stochastic control problem, but the requirement of perfect liquidation in finite time acts as a *nonlocal finite-fuel constraint* on admissible controls. As a result, the corresponding Hamilton-Jacobi-Bellman (HJB) equation involves an infinite initial condition and cannot be solved in general. The case of exponential utility, however, exhibits a close connection to classical mechanics, and we are able to construct a classical solution to our HJB equation by using the variational approach to first-order Hamilton-Jacobi equations described in the monograph by Benton [8]. To identify the optimal strategy, we can then apply verification arguments along with proper localization to deal with the singularity of the value function.

## 2 Statement of main result

We consider a large investor who needs to liquidate a basket  $X_0 = (X_0^1, \dots, X_0^d) \in \mathbb{R}^d$  of shares in  $d$  risky assets by time  $T > 0$ . The investor will thus choose a liquidation strategy that we describe by the number  $X_t^i$  of shares of the  $i^{\text{th}}$  asset held at time  $t$  and that satisfies the boundary condition  $X_T = 0$ . We assume that  $t \mapsto X_t$  is absolutely continuous with derivative  $\dot{X}_t$ , and we parameterize strategies by  $\xi_t := -\dot{X}_t \in \mathbb{R}^d$  i.e.,

$$X_t = X_0 - \int_0^t \xi_s ds.$$

To make the dependence on  $\xi$  explicit, we also write  $X_t^\xi$  in place of  $X_t = X_0 - \int_0^t \xi_s ds$ .

Due to insufficient liquidity, the investor's trading rate  $\xi_t$  is moving the market prices. Several models have been proposed to quantitatively describe this effect. In this note, we consider a  $d$ -dimensional extension of one of the corresponding standard models, namely the model introduced by Almgren [4] (see also Bertsimas and Lo [9] and Almgren and Chriss [5, 6])

for discrete-time precursors of this model), which captures both the permanent and temporary price impacts of large trades, while being sufficiently simple to allow for a mathematical analysis. It therefore has become the basis of several theoretical studies, e.g., [21, 12, 25]. It also demonstrates reasonable properties in real world applications and serves as the basis of many optimal execution algorithms run by practitioners [15, 22, 1, 17].

The permanent price impact for each asset  $i$  accumulates over time and is assumed to be linear:

$$-\sum_{j=1}^d \Gamma^{ij} \xi_t^j dt,$$

where  $\Gamma = (\Gamma^{ij})$  is a symmetric  $d \times d$  matrix. As was observed by Huberman and Stanzl [14], the linearity of the permanent impact rules out the existence of quasi-arbitrage opportunities. The temporary impact vanishes instantaneously and only effects the incremental order  $\xi_t^i$  itself. It is of the form

$$h^i(\xi_t)$$

for a possibly nonlinear function  $h = (h^1, \dots, h^d) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , which needs to satisfy certain mild assumptions (see Assumption 2.1 below). The idealization of instantaneous recovery from the temporary impact is derived from the well-known resilience of stock prices after order placement. It approximates reality reasonably well as long as the time intervals between the physical placement of orders are longer than a few minutes. See, e.g., [10, 19, 26] for empirical studies on resilience in order books and [18, 2, 3] for corresponding market impact models.

When the large investor is not active, it is assumed that the price process  $P_t = (P_t^1, \dots, P_t^d)$  follows a  $d$ -dimensional Bachelier model with linear drift. The resulting stock price dynamics of the  $i^{\text{th}}$  asset are hence given by

$$P_t^i = \tilde{P}_0^i + \sum_{j=1}^m \sigma^{ij} B_t^j + b^i t + \sum_{j=1}^d \Gamma^{ij} (X_t^j - X_0^j) + h^i(\xi_t)$$

for an initial price vector  $\tilde{P}_0 \in \mathbb{R}^d$ , a standard  $m$ -dimensional Brownian motion  $B$  starting at  $B_0 = 0$ , and a (possibly degenerate) volatility matrix  $\sigma = (\sigma^{ij}) \in \mathbb{R}^{d \times m}$ . At first sight, it might seem to be a shortcoming of this model that it allows for negative asset prices. In reality, however, a typical time span  $T$  for the liquidation of a block order will be measured in days or even hours so that  $P_0^i \gg \sqrt{T} \sum_j \sigma^{ij}$ . Therefore negative prices only occur with negligible probability. Moreover, on the scale we are considering, the price process is a random walk on an equidistant lattice and thus perhaps better approximated by an arithmetic rather than, e.g., a geometric Brownian motion.

**Assumption 2.1** *Throughout this note, we assume that*

$$f(x) := x \cdot h(x) = \sum_{i=1}^d x^i h^i(x) \quad \text{for } x \in \mathbb{R}^d \quad (1)$$

*has superlinear growth and is nonnegative, strictly convex, and once continuously differentiable. We also assume that the drift vector  $b$  and the covariance matrix  $\Sigma := \sigma \sigma^\top$  are such that*

$$b \perp \ker \Sigma. \quad (2)$$

In the single asset case a popular choice is, e.g., a temporary impact function that is the sum of a linear and a square root function:  $f(x) = \lambda_1 x^2 + \lambda_2 x^{3/2}$ . A generalization of power law impact to a  $d$ -dimensional setting can be obtained by letting  $f(x) = (x^\top \Lambda x)^\beta$  for a suitable  $d \times d$  matrix  $\Lambda$  and an exponent  $\beta > 1/2$ .

We assume that our strategies  $\xi$  are progressively measurable with respect to a filtration in which  $B$  is a Brownian motion. Strategies also need to be *admissible* in the sense that the resulting position in shares,  $X_t^\xi(\omega)$ , is bounded uniformly in  $t$  and  $\omega$  with upper and lower bounds that may depend on the choice of  $\xi$ . By  $\mathcal{X}(T, X_0)$  we denote the class of all admissible strategies  $\xi$  that liquidate by  $T$  for the initial condition  $X_0$ , i.e., that satisfy  $X_T^\xi = 0$ . The revenues from using such a sales strategy are given by

$$\begin{aligned} \mathcal{R}_T(\xi) &= \int_0^T \xi_t^\top P_t dt \\ &= R_0 + \int_0^T X_t^\top \sigma dB_t + \int_0^T b^\top X_t dt - \int_0^T f(\xi_t) dt, \end{aligned} \quad (3)$$

where

$$R_0 = X_0^\top \tilde{P}_0 - \frac{1}{2} X_0^\top \Gamma X_0. \quad (4)$$

All terms in the previous two equations can be interpreted economically. The first term,  $X_0^\top P_0$  is the face value of the portfolio. The term  $\frac{1}{2} X_0^\top \Gamma X_0$  corresponds to the liquidation costs resulting from the permanent price impact. Due to the linearity of the permanent impact function, it is independent of the choice of the liquidation strategy. The stochastic integral in (3) corresponds to the volatility risk accumulated by selling throughout the interval  $[0, T]$  rather than liquidating the portfolio instantaneously. The integral  $\int_0^T b \cdot X_t dt$  corresponds to the change of portfolio value incurred by the drift. Finally, the integral  $\int_0^T f(\xi_t) dt$  corresponds to the (nonlinear) transaction costs arising from temporary market impact.

We consider here the problem of maximizing the expected utility  $\mathbb{E}[u(\mathcal{R}_T(\xi))]$  of the revenues when  $\xi$  ranges over  $\mathcal{X}(T, X_0)$ , where  $u : \mathbb{R} \rightarrow \mathbb{R}$  is a utility function. When setting up this problem as a stochastic control problem with controlled diffusion process  $\mathcal{R}(\xi)$  and control  $\xi$ , we face the difficulty that the class  $\mathcal{X}(T, X_0)$  of admissible controls depends on both  $X_0$  and  $T$ . To explore some of the effects of this dependence, let us denote by

$$\bar{v}(T, X_0, R_0) := \sup_{\xi \in \mathcal{X}(T, X_0)} \mathbb{E}[u(\mathcal{R}_T(\xi))] \quad (5)$$

the value function of the problem. It depends on the liquidation time,  $T \geq 0$ , the initial portfolio,  $X_0 \in \mathbb{R}^d$ , and  $R_0 \in \mathbb{R}$  in (4). Heuristic arguments suggest that  $\bar{v}$  should satisfy the degenerate Hamilton-Jacobi-Bellman (HJB) equation

$$\bar{v}_T = \frac{X^\top \Sigma X}{2} \bar{v}_{RR} + b^\top X \bar{v}_R - \inf_{\xi \in \mathbb{R}^d} (\xi^\top \nabla_X \bar{v} + \bar{v}_R f(\xi)) \quad (6)$$

with singular initial condition

$$\lim_{T \downarrow 0} \bar{v}(T, X, R) = \begin{cases} u(R) & \text{if } X = 0, \\ -\infty & \text{otherwise.} \end{cases} \quad (7)$$

The singularity in the initial condition (7) reflects the global *fuel constraint*  $\int_0^T \xi_t dt = X_0$  that is required for strategies in  $\mathcal{X}(T, X_0)$ , because it penalizes liquidation that has not been completed in time. Solving this singular Cauchy problem for general utility functions  $u$  seems to be a very difficult problem. But in this note we will show how it—and the corresponding control problem—can be solved in the case of an exponential, or CARA, utility function.

**Theorem 2.2** *For a CARA utility function,  $u(x) = -e^{-\alpha x}$  with risk aversion  $\alpha > 0$ , there exists a unique optimal strategy  $\xi^* \in \mathcal{X}(T, X_0)$ . This strategy  $\xi^*$  is a deterministic function of time. Moreover, the value function  $\bar{v}(T, X, R)$  is a classical solution of the singular Cauchy problem (6), (7).*

To characterize the optimal strategy  $\xi^*$ , let us now focus on the case in which  $\xi$  ranges only over the subclass  $\mathcal{X}_{\text{det}}(T, X_0)$  of *deterministic* strategies in  $\mathcal{X}(T, X_0)$ , i.e., strategies that do not allow for intertemporal updating. In this case,  $\mathcal{R}_T(\xi)$  is normally distributed, and we obtain

$$\mathbb{E}[u(\mathcal{R}_T(\xi))] = -\mathbb{E}[e^{-\alpha \mathcal{R}_T(\xi)}] = -\exp\left(-\alpha \mathbb{E}[\mathcal{R}_T(\xi)] + \frac{\alpha^2}{2} \text{var}(\mathcal{R}_T(\xi))\right). \quad (8)$$

Finding the optimal liquidation strategy is thus reduced to the problem of finding the deterministic strategy  $\xi^*$  that maximizes the mean-variance functional

$$\mathbb{E}[\mathcal{R}_T(\xi)] - \frac{\alpha}{2} \text{var}(\mathcal{R}_T(\xi)) \quad (9)$$

over  $\mathcal{X}_{\text{det}}(T, X_0)$ . This problem of mean-variance optimization has been introduced by Almgren and Chriss [5, 6] for  $d = 1$  and studied extensively since; see, e.g., Almgren [4], Almgren and Lorenz [7], and the references therein. Note that that the variance is weighted here by the factor  $\alpha/2$ , i.e., with *half* of the risk aversion parameter, in contrast to the convention in [6] of using the full risk aversion.

In the case  $d = 1$ , Almgren and Lorenz [7] find that allowing for dynamic updating, i.e., replacing  $\mathcal{X}_{\text{det}}(T, X_0)$  by the entire class  $\mathcal{X}(T, X_0)$  of admissible strategies, can improve the mean-variance performance compared to deterministic strategies. That is, the maximizer of the functional (9) then no longer is a deterministic function of time. For non-deterministic strategies, however, the identity (8) fails, and mean-variance optimization is no longer equivalent to the original problem of maximizing the expected utility of an investor with constant absolute risk aversion. Mean-variance optimization can be regarded as a second-order approximation to the original problem (see, e.g., [13], p. 67, and [9], p. 41). In fact, our main result, Theorem 2.2, shows that in the original problem there is surprisingly *no* added utility from allowing for intertemporal updating of strategies. In contrast to the case of CARA utility considered in the preceding theorem, dynamic strategies *can* improve liquidation performance, but only in the case of a utility function with non-constant absolute risk aversion. Such varying risk aversion is clearly insufficiently captured by the mean-variance approximation, and optimal strategies are no longer deterministic. It is shown in [23] and [24] that for an infinite liquidation horizon,  $T = \infty$ , deterministic strategies maximize expected utility if and only if the underlying utility function is of CARA type. We strongly believe that this result carries over to the case of a finite liquidation horizon  $T < \infty$ .

Clearly, the maximization of the mean-variance functional (9) over deterministic strategies is equivalent to the minimization of the action functional

$$\int_0^T \mathcal{L}(X_t, \dot{X}_t) dt \quad (10)$$

over curves with  $X_0$  and  $X_T = 0$  given, where the Lagrangian  $\mathcal{L}$  is given by

$$\mathcal{L}(q, p) = \frac{\alpha}{2} q^\top \Sigma q - b^\top q + f(-p), \quad q, p \in \mathbb{R}^d. \quad (11)$$

Minimizing this action functional is a classical problem in the calculus of variations. In particular,  $X^* := X^{\xi^*}$  can be obtained as a solution of the Euler-Lagrange equation

$$\alpha \Sigma X_t = b + \nabla^2 f(-\dot{X}_t) \ddot{X}_t \quad (12)$$

with boundary conditions  $X_0^* = X_0$  and  $X_T^* = 0$ , provided that  $f$  is even *twice continuously differentiable*. However, as we have seen in the Examples subsequent to Assumption 2.1, the requirement  $f \in C^2(\mathbb{R}^d)$  is too strong for many relevant examples. Alternatively, one can try to characterize  $X^{\xi^*}$  by the corresponding Hamilton equations

$$\begin{aligned} \dot{X}_t &= \mathcal{H}_p(X_t, p(t)) \\ \dot{p}(t) &= -\mathcal{H}_q(X_t, p(t)), \end{aligned} \quad (13)$$

where

$$\mathcal{H}(q, p) = -\frac{\alpha}{2} q^\top \Sigma q + b^\top q + f^*(-p)$$

is the Hamiltonian for  $\mathcal{L}$  and  $f^*(z) = \sup_x (x^\top z - f(x))$  denotes the Fenchel-Legendre transform of  $f$ . It turns out that, owing to the particular form of  $\mathcal{L}$ , this approach is possible even in the general case in which we do not require more regularity on  $f$  than being  $C^1(\mathbb{R}^d)$ :

**Theorem 2.3** *The optimal  $X^* = X^{\xi^*}$  is continuously differentiable in  $t$  and satisfies the Hamilton equations*

$$\begin{aligned} \dot{X}_t &= -\nabla f^*(-p(t)) \\ \dot{p}(t) &= \alpha \Sigma X_t - b, \end{aligned} \quad (14)$$

with initial conditions  $X_0^* = X_0$  and  $p(0) = -\nabla f(-\dot{X}_0^*)$ . Conversely, if  $\xi \in \mathcal{X}_{\text{det}}(T, X_0)$  is such that  $X^\xi$  a.e. satisfies (14) for some function  $p(t)$ , then  $\xi = \xi^*$  almost everywhere.

**Example 2.4 (Linear temporary impact)** Linear temporary impact corresponds to the function

$$f(x) = x^\top \Lambda x$$

for a suitable  $d \times d$  matrix  $\Lambda$ . Since then (12) or (14) are inhomogeneous linear ODEs, the optimal strategy  $\xi^* \in \mathcal{X}(T, X_0)$  is the sum  $\xi^0 + \xi^1$  of the optimal strategy  $\xi^0 \in \mathcal{X}(T, 0)$ , which corresponds to the optimal ‘round trip’ exploiting the market drift, and the optimal strategy  $\xi^1 \in \mathcal{X}(T, X_0)$  for the market model with drift  $b = 0$ . This fact was already observed by Almgren

and Chriss [6] in their discrete-time model. In dimension  $d = 1$ , the matrices  $\Sigma$  and  $\Lambda$  have the single entries  $\sigma^2$  and  $\lambda$ , respectively, and by setting

$$\kappa := \sqrt{\frac{\alpha\sigma^2}{2\lambda}}$$

we obtain the explicit formulas

$$\xi_t^0 = -b \cdot \frac{e^{\kappa(T-t)} - e^{\kappa t}}{\sqrt{2\lambda\alpha\sigma^2}(1 + e^{\kappa T})}, \quad X_t^{\xi^0} = \frac{b(e^{\kappa(T-t)} - 1)(e^{\kappa t} - 1)}{\alpha\sigma^2(1 + e^{\kappa T})},$$

and

$$\xi_t^1 = \kappa X_0 \cdot \frac{\cosh(\kappa(T-t))}{\sinh(\kappa T)}, \quad X_t^{\xi^1} = X_0 \cdot \frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)}.$$

### 3 Proofs

For  $\xi \in \mathcal{X}(T, X_0)$ , we define the stochastic processes  $X_t^\xi := X_0 - \int_0^t \xi_s ds$  and

$$R_t^\xi := R_0 + \int_0^t X_s^\xi \cdot \sigma dB_s + \int_0^t b \cdot X_s^\xi ds - \int_0^t f(\xi_s) ds, \quad 0 \leq t \leq T.$$

If we take  $R_0 = \tilde{P}_0^\top X_0 - \frac{1}{2} X_0^\top \Gamma X_0$ , then  $R_T^\xi$  coincides with the revenues  $\mathcal{R}_T(\xi)$  of  $\xi$ . But  $R_0$  can also incorporate an initial cash position or revenues that carry over from the ‘past’ after an application of the Markov property.

In proving our main results we will simplify our formulas by dropping the first minus sign in the utility function  $u(x) = -e^{-\alpha x}$ . Thus, our goal is to identify the value function

$$v(T, X_0, R_0) := \inf_{\xi \in \mathcal{X}(T, X_0)} \mathbb{E}[e^{-\alpha R_T^\xi}].$$

with the function

$$w(T, X_0, R_0) := \exp \left[ -\alpha R_0 + \alpha \inf_{\xi \in \mathcal{X}_{\text{det}}(T, X_0)} \int_0^T L(X_t^\xi, \xi_t) dt \right], \quad (15)$$

where  $L$  is the Lagrangian

$$L(q, p) = \frac{\alpha}{2} q^\top \Sigma q - b^\top q + f(p), \quad q, p \in \mathbb{R}^d.$$

Note that  $L(q, p) = \mathcal{L}(q, -p)$ , where  $\mathcal{L}$  is as in (10). To this end, we use stochastic control methods. More precisely, our proof is based on the observation that  $w$  is a classical solution of the following HJB equation with appropriate singular initial condition. Note that  $w(T, X_0, R_0)$  is clearly a smooth function in  $R_0$ .

**Lemma 3.1** *The function  $w(T, X, R)$  is continuously differentiable in  $T > 0$ ,  $X \in \mathbb{R}^d$  and satisfies the HJB equation*

$$w_T = \frac{1}{2} X^\top \Sigma X w_{RR} + b^\top X w_R - \sup_{\xi \in \mathbb{R}^d} (\xi^\top \nabla_X w + w_R f(\xi)) \quad (16)$$

with singular initial condition

$$\lim_{T \downarrow 0} w(T, X, R) = \begin{cases} e^{-\alpha R} & \text{if } X = 0, \\ +\infty & \text{if } X \neq 0. \end{cases} \quad (17)$$

**Proof:** Let

$$S(T, X_0) := \inf_{\xi \in \mathcal{X}_{\det}(T, X_0)} \int_0^T L(X_t^\xi, \xi_t) dt.$$

Our aim is to show that  $S$  solves the Hamilton-Jacobi equation

$$S_T(T, X) + H(X, \nabla_X S(T, X)) = 0, \quad T > 0, X \in \mathbb{R}^d, \quad (18)$$

where

$$H(q, p) = -\frac{1}{2}\alpha q^\top \Sigma q + b^\top q + f^*(p), \quad q, p \in \mathbb{R}^d, \quad (19)$$

is the Hamiltonian corresponding to the Lagrangian  $L$ .

To see that  $S(T, X)$  is continuously differentiable and satisfies (18), we now apply Theorem 7.1 in Chapter II of Benton [8]. First, note that by taking  $B := \{(0, 0)\} \subset \mathbb{R} \times \mathbb{R}^d$  and  $g : B \rightarrow \mathbb{R}$  as  $g(0, 0) := 0$ ,  $S$  can be written as

$$S(T, X_0) = \inf \left\{ g(s, y) + \int_s^T L(Y_t, \dot{Y}_t) dt \mid (s, y) \in B \right\},$$

where the infimum is taken over all absolutely continuous curves  $Y$  such that  $Y_s = y$  and  $Y_T = X_0$ . Next, we note that this class can be replaced by the smaller class of Lipschitz continuous curves  $Y$ . Indeed, an absolutely continuous curve  $Y$  can easily be approximated by Lipschitz curves  $Y^n$  such that  $Y_s^n = y$ ,  $Y_T^n = 0$ , and  $\int_0^T L(Y_t^n, \dot{Y}_t^n) dt \rightarrow \int_0^T L(Y_t, \dot{Y}_t) dt$ . Thus, we are in the context of Theorem 7.1 in Chapter II of Benton [8]. Let us check next that its conditions (H1)–(H6) and (D1)–(D4) are satisfied.

By Theorem 26.6 in Rockafellar [20],  $f^*$  is strictly convex, continuously differentiable, and satisfies the superlinear growth condition. Hence, the same properties are satisfied by  $p \mapsto H(q, p)$  for any  $q$ , and this establishes conditions (H1), (H2), and (H3). Next,

$$\begin{aligned} p\nabla_p H(q, p) - H(q, p) &= p\nabla f^*(p) - f^*(p) + \frac{1}{2}\alpha q^\top \Sigma q - b^\top q \\ &= f(\nabla f^*(p)) + \frac{1}{2}\alpha q^\top \Sigma q - b^\top q, \end{aligned}$$

due to Theorems 26.5 and 26.6 in [20]. Note that (2) implies that there exists a constant  $M$  such that

$$|b^\top q| \leq \frac{1}{4}\alpha q^\top \Sigma q + M. \quad (20)$$

Since  $f \geq 0$ , there is hence a constant  $C$  such that for all  $q \in \mathbb{R}^d$

$$|\nabla_q H(q, p)| \leq C \left( p\nabla_p H(q, p) - H(q, p) + 1 \right),$$

and this is condition (H4). Moreover,

$$p\nabla_p H(q, p) - H(q, p) \geq \frac{1}{2}\alpha q^\top \Sigma q - b^\top q \geq -M,$$

which is condition (H5). Next, it is clear that both  $H(q, p)$  and  $|\nabla_p H(q, p)|$  can be bounded from above by an increasing function of  $|p|$ , i.e., (H6) holds. Conditions (D1) to (D4) are void in our situation. Therefore, Theorem 7.1 in Chapter II of Benton [8] is applicable, and  $S$  satisfies (18).

A straightforward computation now shows that

$$w(T, X, R) = \exp(-\alpha R + \alpha S(T, X)) \quad (21)$$

solves

$$w_T = \frac{X^\top \Sigma X}{2} w_{RR} + b^\top X w_R + w_R f^* \left( \frac{-\nabla_X w}{w_R} \right),$$

and this equation is equivalent to (16).

As for the initial condition (17), it is clear that  $S$  satisfies  $S(T, 0) = 0$  for all  $T$ . Moreover, if  $Y$  is any Lipschitz curve such that  $Y_0 = 0$  and  $Y_T = X \neq 0$ , then

$$\int_0^T L(Y_t, \dot{Y}_t) dt = \int_0^T \left( \frac{1}{2} \alpha Y_t^\top \Sigma Y_t - b \cdot Y_t \right) dt + \int_0^T f(\dot{Y}_t) dt \geq -TM + Tf\left(\frac{X}{T}\right),$$

by Jensen's inequality and (20). Hence,  $S(T, X) \geq -MT + Tf(X/T)$ , and this expression blows up as  $T \downarrow 0$  by to our superlinear growth condition on  $f$ .  $\square$

**Lemma 3.2** *For every  $X_0 \in \mathbb{R}^d$  there exists an a.e. unique curve  $\xi^*$  minimizing the action functional  $\int_0^T L(X_t^\xi, \xi_t) dt$  over  $\xi \in \mathcal{X}_{\det}(T, X_0)$ . Moreover,  $X_t^{\xi^*}$  is Lipschitz continuous in  $t$ .*

**Proof:** We have already argued in the proof of Lemma 3.1 that the infimum of  $\int_0^T L(X_t^\xi, \xi_t) dt$  taken over  $\xi \in \mathcal{X}_{\det}(T, X_0)$  is the same as the infimum taken over the subset of  $\xi \in \mathcal{X}_{\det}(T, X_0)$  for which  $X^\xi$  is Lipschitz continuous. Existence of a minimizer in this class follows from Lemma 7.3 in Chapter II of [8], because its conditions are a subset of the conditions of Theorem 7.1 in Chapter II of [8], which are satisfied due to the arguments in the proof of Lemma 3.1. The uniqueness of the minimizer follows from the strict convexity of  $(q, p) \mapsto L(q, p)$ .  $\square$

From now on we will refer to the strategy  $\xi^*$  obtained in the preceding Lemma as the *optimal strategy in  $\mathcal{X}_{\det}(T, X_0)$* . We then have

$$w(T, X_0, R_0) = \min_{\xi \in \mathcal{X}_{\det}(T, X_0)} \mathbb{E}[e^{-\alpha R_T^\xi}] = \mathbb{E}[e^{-\alpha R_T^{\xi^*}}]. \quad (22)$$

Let us next introduce the sets

$$\mathcal{X}^{(K)}(T, X_0) := \{ \xi \in \mathcal{X}(T, X_0) \mid |X_t^\xi| \leq K \text{ for all } t \}$$

and the value functions

$$v^{(K)}(T, X, R) := \inf_{\xi \in \mathcal{X}^{(K)}(T, X)} \mathbb{E}[e^{-\alpha R_T^\xi}].$$

Note that  $\mathcal{X}^{(K)}(T, X)$  is empty for small  $K < |X|$  and that  $\mathcal{X}(T, X) = \bigcup_K \mathcal{X}^{(K)}(T, X)$ . Moreover,  $\xi^* \in \mathcal{X}^{(K)}(T, X)$  for some  $K$ , because  $X^{\xi^*}$  is continuous and deterministic. We continue with the following a priori estimate.

**Lemma 3.3** *For  $|X| \leq K$ ,*

$$v^{(K)}(T, X, R) \geq w(T, X, R) \cdot e^{-2\alpha|b|KT - \alpha^2 X^\top \Sigma X T / 6}.$$

**Proof:** Suppose that  $|X_0| \leq K$  and  $\xi \in \mathcal{X}^{(K)}(T, X_0)$  is such that  $\mathbb{E}[e^{-\alpha R_T^\xi}]$  is finite. We then have

$$\infty > \mathbb{E}[e^{-\alpha R_T^\xi}] \geq e^{-\alpha \mathbb{E}[R_T^\xi]}. \quad (23)$$

Since  $X^\xi$  is bounded, we have

$$\mathbb{E}[R_T^\xi] = R_0 + \int_0^T \mathbb{E}[b^\top X_t^\xi - f(\xi_t)] dt \leq R_0 + TK|b| - \int_0^T \mathbb{E}[f(\xi_t)] dt.$$

By (23), it follows that  $\int_0^T \mathbb{E}[f(\xi_t)] dt$  is finite, and so  $\bar{x}_t := \mathbb{E}[\xi_t]$  is well defined and integrable, due to our assumptions on  $f$ . Hence,  $\bar{x}$  belongs to  $\mathcal{X}_{\det}(T, X_0)$ . Applying Jensen's inequality twice yields

$$\int_0^T \mathbb{E}[f(\xi_t)] dt \geq \int_0^T f(\bar{x}_t) dt \geq Tf\left(\frac{X_0}{T}\right).$$

Therefore,

$$\mathbb{E}[e^{-\alpha R_T^\xi}] \geq e^{-\alpha \mathbb{E}[R_T^\xi]} \geq e^{-\alpha R_0 - \alpha TK|b| + \alpha Tf(X_0/T)} =: \underline{v}(T, X_0, R_0),$$

and in turn  $v^{(K)}(T, X_0, R_0) \geq \underline{v}(T, X_0, R_0)$ .

On the other hand, the constant strategy  $\hat{x}_t = X_0/T$  belongs to  $\mathcal{X}_{\det}(T, X_0) \cap \mathcal{X}^{(K)}(T, X_0)$ , and so (22) yields that

$$\begin{aligned} w(T, X_0, R_0) &\leq \mathbb{E}[e^{-\alpha R_T^{\hat{x}}}] = \underline{v}(T, X_0, R_0) \cdot e^{\alpha T(K|b| - \frac{1}{2}b^\top X_0)} \cdot \mathbb{E}\left[e^{-\alpha \int_0^T \frac{(T-t)}{T} X_0^\top \sigma dB_t}\right] \\ &= \underline{v}(T, X_0, R_0) \cdot e^{\alpha T(K|b| - \frac{1}{2}b^\top X_0)} \cdot e^{\alpha^2 X_0^\top \Sigma X_0 T/6} \\ &\leq v^{(K)}(T, X_0, R_0) \cdot e^{2\alpha TK|b| + \alpha^2 X_0^\top \Sigma X_0 T/6}. \end{aligned}$$

This proves the lemma.  $\square$

In the next step, we will use a verification argument to identify  $w$  with the modified value function

$$v_0(T, X, R) := \inf_{\xi \in \mathcal{X}_0(T, X)} \mathbb{E}[e^{-\alpha R_T^\xi}] \quad (24)$$

that is based on

$$\mathcal{X}_0(T, X) := \left\{ \xi \in \mathcal{X}(T, X) \mid \int_0^T f(\xi_t) dt \text{ is bounded} \right\}.$$

**Lemma 3.4** *We have  $v_0(T, X, R) = w(T, X, R)$  and the unique minimizing strategy in (24) is given by the  $\xi^*$ .*

**Proof:** The inequality  $w \geq v_0$  is obvious from (22). To prove the converse inequality, take  $K > |X_0|$  and let  $\xi \in \mathcal{X}_0(T, X_0) \cap \mathcal{X}^{(K)}(T, X_0)$  be a control process. For  $0 < t < T$ , Itô's formula yields that

$$\begin{aligned} w(T-t, X_t^\xi, R_t^\xi) - w(T, X_0, R_0) &= \int_0^t w_R(T-s, X_s^\xi, R_s^\xi) (X_s^\xi)^\top \sigma dB_s \\ &\quad - \int_0^t \left[ f(\xi_s) w_R + \xi_s^\top \nabla_X w + w_T - b^\top X_s^\xi w_R - \frac{1}{2} (X_s^\xi)^\top \Sigma X_s^\xi w_{RR} \right] (T-s, X_s^\xi, R_s^\xi) ds. \end{aligned} \quad (25)$$

By (16), the latter integral is nonpositive, and by noting that  $w_R = -\alpha w$  we obtain

$$w(T, X_0, R_0) \leq w(T-t, X_t^\xi, R_t^\xi) + \alpha \int_0^t w(T-s, X_s^\xi, R_s^\xi) (X_s^\xi)^\top \sigma dB_s. \quad (26)$$

We will show next that the stochastic integral in (26) is a true martingale. To this end, observe first that, for some constant  $C_1$  depending on  $R_0$ , on the bound on  $\int_0^T f(\xi_t) dt$ , and on the upper bound  $K$  for  $|X^\xi|$ ,

$$R_s^\xi := R_0 + \int_0^s (X_r^\xi)^\top \sigma dB_r + \int_0^s b^\top X_r^\xi dr - \int_0^s f(\xi_r) dr \geq -C_1 \left(1 + \sup_{r \leq TK^2|\Sigma|} |W_r|\right),$$

where  $W$  denotes the DDS-Brownian motion of the continuous martingale  $\int_0^t (X_s^\xi)^\top \sigma dB_s$  and  $|\Sigma|$  is the operator norm of  $\Sigma$ . Hence, by (15), for a constant  $C_2$  depending on  $K$  and the upper bound of  $\int_0^T f(\xi_t) dt$ ,

$$\begin{aligned} w(T-s, X_s^\xi, R_s^\xi) &\leq \exp \left( \alpha C_1 \left(1 + \sup_{r \leq TK^2|\Sigma|} |W_r|\right) + \alpha \int_s^T L(X_t^\xi, \xi_t) dt \right) \\ &\leq \exp \left( \alpha C_1 \left(1 + \sup_{r \leq TK^2|\Sigma|} |W_r|\right) + C_2 \right). \end{aligned} \quad (27)$$

Since  $\sup_{s \leq TK^2|\Sigma|} |W_s|$  has exponential moments of all orders, the martingale property of the stochastic integral in (26) follows.

Taking expectations in (26) thus yields

$$w(T, X_0, R_0) \leq \mathbb{E}[w(T-t, X_t^\xi, R_t^\xi)]. \quad (28)$$

Using the fact that

$$v^{(K)}(T-t, X_t^\xi, R_t^\xi) \leq \mathbb{E}[e^{-\alpha R_T^\xi} | \mathcal{F}_t],$$

Lemma 3.3 then gives that for  $C_3 := 2\alpha|b|K + \alpha^2 K^2 |\Sigma|/6$ ,

$$w(T, X, R) \leq e^{C_3(T-t)} \mathbb{E}[v^{(K)}(T-t, X_t^\xi, R_t^\xi)] \leq e^{C_3(T-t)} \mathbb{E}[e^{-\alpha R_T^\xi}].$$

Sending  $t \uparrow T$  and taking the infimum over  $\xi \in \mathcal{X}_0^{(K)}(T, X_0)$  and then over  $K \geq |X_0|$  yields  $w \leq v_0$ .

Uniqueness of the optimal strategy follows from the strict convexity of  $\xi \mapsto \mathbb{E}[e^{-\alpha R_T^\xi}]$ .  $\square$

**Proof of Theorem 2.2:** We first show that  $w \leq v$ . To this end, let  $\xi \in \mathcal{X}(T, X_0)$  be given such that

$$\mathbb{E}[e^{-\alpha R_T^\xi}] < \infty. \quad (29)$$

We define for  $k = 1, 2, \dots$

$$\tau_k := \inf \left\{ t \geq 0 \mid \int_0^t f(\xi_s) ds \geq k \text{ or } (T-t)f\left(\frac{X_t^\xi}{T-t}\right) \geq k \right\} \wedge T.$$

Let  $K$  be an upper bound for  $|X^\xi|$  and define  $C := 2\alpha|b|K + \alpha^2K^2|\Sigma|/6$ . Conditioning on  $\mathcal{F}_{\tau_k}$  and applying Lemma 3.3 yields that

$$\begin{aligned} \mathbb{E}[e^{-\alpha R_T^\xi}; \tau_k < T] &\geq \mathbb{E}[v^{(K)}(T - \tau_k, X_{\tau_k}^\xi, R_{\tau_k}^\xi); \tau_k < T] \\ &\geq \mathbb{E}[w(T - \tau_k, X_{\tau_k}^\xi, R_{\tau_k}^\xi) \cdot e^{-C(T-\tau_k)}; \tau_k < T] \\ &\geq e^{-CT} \cdot \mathbb{E}[w(T - \tau_k, X_{\tau_k}^\xi, R_{\tau_k}^\xi); \tau_k < T] \\ &\geq 0. \end{aligned}$$

Since  $\mathbb{E}[e^{-\alpha R_T^\xi}; \tau_k < T]$  tends to zero as  $k \uparrow \infty$  due to (29), we conclude that also

$$\mathbb{E}[w(T - \tau_k, X_{\tau_k}^\xi, R_{\tau_k}^\xi); \tau_k < T] \longrightarrow 0. \quad (30)$$

Moreover,

$$\mathbb{E}[e^{-\alpha R_T^\xi}; \tau_k = T] = \mathbb{E}[w(0, 0, R_T^\xi); \tau_k = T] = \mathbb{E}[w(T - \tau_k, X_{\tau_k}^\xi, R_{\tau_k}^\xi); \tau_k = T].$$

We may thus conclude that

$$\begin{aligned} \mathbb{E}[e^{-\alpha R_T^\xi}] &= \mathbb{E}[e^{-\alpha R_T^\xi}; \tau_k < T] + \mathbb{E}[e^{-\alpha R_T^\xi}; \tau_k = T] \\ &\geq \mathbb{E}[w(T - \tau_k, X_{\tau_k}^\xi, R_{\tau_k}^\xi)] - (1 - e^{-CT}) \cdot \mathbb{E}[w(T - \tau_k, X_{\tau_k}^\xi, R_{\tau_k}^\xi); \tau_k < T]. \end{aligned}$$

Now define  $\xi^{(k)}$  as  $\xi$  up to time  $\tau_k$ , and for  $\tau_k < t \leq T$  we let  $\xi^{(k)}$  be the  $\mathcal{F}_{\tau_k}$ -measurable strategy from Lemma 3.2 that optimally liquidates the amount  $X_{\tau_k}^\xi$  in the remaining time,  $T - \tau_k$ . Then  $\xi^{(k)}$  belongs to  $\mathcal{X}_0(T, X_0)$ , and Lemma 3.4 yields

$$\mathbb{E}[w(T - \tau_k, X_{\tau_k}^\xi, R_{\tau_k}^\xi)] = \mathbb{E}[e^{-\alpha R_T^{\xi^{(k)}}}] \geq w(T, X_0, R_0).$$

Therefore

$$\mathbb{E}[e^{-\alpha R_T^\xi}] \geq w(T, X, R) - (1 - e^{-CT}) \cdot \mathbb{E}[w(T - \tau_k, X_{\tau_k}^\xi, R_{\tau_k}^\xi); \tau_k < T].$$

Using (30) now yields  $w \leq v$  and, in view of (22),  $v = w$  and the optimality of  $\xi^*$  in  $\mathcal{X}(T, X_0)$ .

Finally, uniqueness of the optimal strategy follows again from the strict convexity of  $\xi \mapsto \mathbb{E}[e^{-\alpha R_T^\xi}]$ .  $\square$

**Proof of Theorem 2.3:** Let us first show that  $X^{\xi^*}$  is continuously differentiable on  $[0, T]$ . By the arguments in the proof of Lemma 3.4, the finite-variation term in (25) vanishes for all  $t$  if we take  $\xi = \xi^*$ . That is, for a.e.  $s \in ]0, T[$ ,

$$0 = \left[ f(\xi_s^*)w_R + \xi_s^* \cdot \nabla_X w + w_T - b \cdot X_s^{\xi^*} w_R - \frac{1}{2} (X_s^{\xi^*})^\top \Sigma X_s^{\xi^*} w_{RR} \right] (T - s, X_s^{\xi^*}, R_s^{\xi^*}).$$

By using (21) with the function  $S(T, X)$  defined in the proof of Lemma 3.1, we see that the preceding identity is equivalent to

$$\begin{aligned} 0 &= S_T(T - s, X_s^{\xi^*}) - \frac{\alpha}{2} (X_s^{\xi^*})^\top \Sigma X_s^{\xi^*} + b^\top X_s^{\xi^*} + \nabla_X S(T - s, X_s^{\xi^*}) \xi_s^* - f(\xi_s^*) \\ &= -f^*(\nabla_X S(T - s, X_s^{\xi^*})) + \nabla_X S(T - s, X_s^{\xi^*}) \xi_s^* - f(\xi_s^*) \end{aligned}$$

for a.e.  $s \in ]0, T[$ , where we have used (18) in the second step. It follows from Theorems 23.5 and 26.5 in [20] that

$$-\dot{X}_s^{\xi^*} = \xi_s^* = \nabla f^*(\nabla_X S(T-s, X_s^{\xi^*})) \quad \text{for a.e. } s \in ]0, T[.$$

Clearly, the right-hand side is a continuous function of  $s \in [0, T[$ .

To see that  $\dot{X}_s^{\xi^*}$  admits a limit for  $s \uparrow T$ , observe that  $\hat{\xi}_t := -\xi_{T-t}^*$  belongs to  $\mathcal{X}_{\det}(T, -X_0)$  and is the minimizer of  $\int_0^T \widehat{L}(X_t^\xi, \xi_t) dt$  over this class (we take here  $X_0^\xi = -X_0$ ), where

$$\widehat{L}(q, p) := \frac{\alpha}{2} q^\top \Sigma q - \widehat{b}^\top q + f(p)$$

for  $\widehat{b} = b - \alpha X_0^\top \Sigma$ . Hence, by the first step of this proof,  $X_t^{\hat{\xi}} = -X_0 + \int_0^t \widehat{\xi}_s ds = X_{T-t}^{\xi^*} - X_0$  is continuously differentiable in  $t \in [0, T[$ . In particular,  $\lim_{t \downarrow 0} \dot{X}_t^{\hat{\xi}} = \lim_{s \uparrow T} \dot{X}_s^{\xi^*}$  exists.

Now we analyze the Hamilton equations (14). Let  $\xi \in \mathcal{X}_{\det}(X_0, T)$  be bounded and let  $Z : [0, T] \rightarrow \mathbb{R}^d$  be Lipschitz continuous with  $Z_0 = 0 = Z_T$ . Then

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_0^T L(X_t^\xi + \varepsilon Z_t, \xi_t - \varepsilon \dot{Z}_t) dt = \int_0^T \left[ Z_t^\top (\alpha \Sigma X_t^\xi - b) - \dot{Z}_t^\top \nabla f(\xi_t) \right] dt. \quad (31)$$

Letting  $p(t) := p_0 + \int_0^t (\alpha \Sigma X_s^\xi - b) ds$ , we see that the expression in the display vanishes for all  $Z$  if and only if there exists some  $p_0$  such that  $p(t) = -\nabla f(\xi_t)$  holds for a.e.  $t$ . By Theorem 26.6 in [20] the latter identity is equivalent to  $\xi_t = \nabla f^*(-p(t))$  and hence to the fact that  $(X^\xi, p)$  satisfies the Hamilton equations (14) for a.e.  $t$ . Since  $\xi^*$  is optimal and hence a critical point of the action functional, the Hamilton equations must be satisfied in this case.

Conversely, if another  $\xi$  is such that  $X^\xi$  satisfies the Hamilton equations (14) for some function  $p$  and for a.e.  $t$ , then it readily follows that both  $X^\xi$  and  $p$  are continuously differentiable and in particular Lipschitz continuous. Hence, (31) applies and  $\xi$  corresponds to a critical point of the action functional. But  $L(p, q)$  is strictly convex in  $p$  and  $q$ , and so the action functional is strictly convex and therefore has at most one critical point.  $\square$

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