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Wielandt's proof of the exponent inequality for primitive nonnegative matrices

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Dedicated to Annemarie Wielandt for her help in preserving a mathematical legacy

Abstract

The proof of the exponent inequality found in Wielandt's unpublished diaries of a result announced without proof in his well known paper on nonnegative irreducible matrices. A facsimile, a transcription, a translation and a commentary are presented.
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In his famous paper [3] on nonnegative irreducible matrices published in 1950, Wielandt announced an inequality for the exponent of a primitive matrix and gave an example to show that it was sharp. However he did not give a proof. Recently the proof has been found in his unpublished mathematical diaries. This note contains a facsimile, transcription and English translation of the proof followed by a commentary on it.

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Ende 11
Woche Sept. 49, nach Köln (20.9.)

Nichtnegative Matrizen.

Ist A unzerlegbar, ≥ 0, und ist die Maximalwurzel r die einzige vom maximalen Betrag, so ist für genügend grosses p $A^p > 0$.

Frobenius zeigt 1912: Ist A unzerl., $a_{11} > 0$, so ist in A^{n-1} die erste Zeile u. Spalte positiv, also ist $A^{2n-2} > 0$. Daher ist, wenn A unzerlegbar und r die einz. Wurzel vom max. Betrag (d.h. A „primitiv"), da ferner nach ~~Cayley~~ $\widehat{(rE-A)} > 0$ auch $A\widehat{(rE-A)} > 0$ und Linearverbindung von $A, \dots, A^n$ ist, folgert Frob., dass spätestens für $m+1$ $a_{11}^{(m)} > 0$ ist. Er fasst nicht beides zusammen, man würde sonst bekommen: Ist A primitiv, so $A^{n(2n-2)} > 0$.

Satz: Der genaue Exponent ist $n^2-2n+2 = p$: " " , " $A^p > 0$.

Bew: (1) Gibt es zu jedem $\alpha = 1, \dots, n$ ein l aus $1 \leq l \leq n-1$ mit $a_{\alpha\alpha}^{(l)} > 0$, so hat $A^{l(n-1)}$ die α-te Spalte positiv, also auch $A^{(n-1)^2}$, also auch A^p.

(2) Zu einem α, etwa $\alpha = 1$, sei $a_{11} = a_{11}^{(2)} = \dots = a_{11}^{(n-1)} = 0$. Dann ist $a_{11}^{(n)} \neq 0$ (z.B. nach Cayley), d.h. es gibt n Faktoren $a_{1\rho} a_{\rho\sigma} a_{\sigma\tau} \cdots a_{\omega 1} \neq 0$, aber nicht weniger. Aus dem letzten Grund sind die n Indizes $1, \rho, \sigma, \tau, \dots, \omega$ alle verschieden, also bei passender Anordnung genau $= 1, 2, \dots, n$: $A \geq$ [illegible]. ~~[illegible]~~ Hieraus folgt, wenn eine Potenz A^q eine positive Spalte hat, dass unter $A^q, A^{q+1}, \dots, A^{q+(n-1)}$ ~~[illegible]~~ die α-te Spalte einmal > 0 ist, also ist jede Spalte von $A^{q+(n-1)} > 0$. Nun zwei Fälle:

(a) einmal ist $\mathrm{Sp}\, A^\lambda \neq 0$ für ein λ aus $1 \leq \lambda \leq n-2$. Dann einmal $a_{\alpha\alpha}^{(\lambda)} > 0$, also $A^{\lambda(n-1)}$ hat positive α-te Spalte, also $A^{\lambda(n-1)+(n-1)} > 0$, $A^{(\lambda+1)(n-1)} > 0$, $A^{(n-1)^2} > 0$, $A^p > 0$

(b) $\mathrm{Sp}\, A = \mathrm{Sp}\, A^2 = \dots = \mathrm{Sp}\, A^{n-2} = 0$. Dann Cayley (da $S_1, S_2, \dots, S_{n-2} = 0$)
$A^n = aE + bA$, hierin $a, b > 0$, da sonst imprimitiv (s.o.)
dann $A^{n(n-2)+2} \sim A^2(E+A)^{n-2} \sim A^2 + A^3 + \dots + A^{n-1} + A^n \sim E + A + A^2 + \dots + A^{n-1} > 0$.

Für $A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$ gilt $A^n = E + A$ ~~und daher~~ $A^{p-1} \not> 0$, da
$A^{p-1} = A(E+A)^{n-2} \sim$ ~~[illegible]~~ $\dots + A^{n-1}$ $a_{11}^{(n-1)} = 0$ hat.

Transcription of Wielandt's proof[1]
From HW Diary VII, p. 11 and p. 10

Nichtnegative Matrizen.

Ist A unzerlegbar, $\geqslant 0$, und ist die Maximalwurzel r die einzige vom maximalen Betrag, so ist für genügend grosses p $A^p > 0$.

Frobenius zeigt 1912: Ist A unzerl., $a_{11} > 0$, so ist in A^{n-1} die erste Zeile u. Spalte positiv, also ist $A^{2n-2} > 0$. Daher ist, wenn A unzerlegbar und r die einzige Wurzel vom max. Betrag (dh. A "primitive") Da ferner nach ~~Cayley~~ $(\widehat{rE - A}) > 0$ auch $A(\widehat{rE - A}) > 0$ und Linearverbindung von $A, \ldots, A^n$ ist, folgert Frob., dass spätestens für $m = n$ $a_{11}^{(m)} > 0$ ist. Er fasst nicht beides zusammen, man würde so bekommen: Ist A primitiv, so $A^{n(2n-2)} > 0$.

Satz: Der genaue Exponent ist $n^2 - 2n + 2 = p$: Ist A primitiv, so $A^p > 0$.

Bew.: (1) Gibt es zu jedem $\alpha = 1, \ldots, n$ ein ℓ aus $1 \leqslant \ell \leqslant n-1$ mit $a_{\alpha\alpha}^{(\ell)} > 0$, so hat $A^{\ell(n-1)}$ die α-te Spalte positive, also auch $A^{(n-1)^2}$, also auch A^p.

(2) Zu einem α, etwa $\alpha = 1$, sei $a_{11} = a_{11}^{(2)} = \cdots = a_{11}^{(n-1)} = 0$. Dann ist $a_{11}^{(n)} \neq 0$ (zB, nach Cayley), dh. es gibt n Faktoren $a_{1\rho}a_{\rho\sigma}a_{\sigma\tau}\cdots a_{\omega 1} \neq 0$, aber nicht weniger. Aus dem letzten Grund sind die n Indizes $1, \rho, \sigma, \tau, \ldots \omega$ alle verschieden, also bei passender Anordnung genau $= 1, 2, \ldots, n$;

$$A \gtrsim \begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ 1 & & & & \end{pmatrix}$$

hieraus folgt, wenn eine Potenz A^q eine positive Spalte hat, dass unter $A^q, A^{q+1}, \ldots, A^{q+(n-1)}$ die α-te Spalte einmal > 0 ist, also ist *jede* Spalte von $A^{q+(n-1)} > 0$. Nun zwei Fälle:

(a) einmal ist Sp $A^\lambda \neq 0$, für ein λ aus $1 \leqslant \lambda \leqslant n-2$. Dann einmal $a_{\alpha\alpha}^{(\lambda)} > 0$, also $A^{\lambda(n-1)}$ hat positive α-te Spalte, also $A^{\lambda(n-1)+(n-1)} > 0$, $A^{(\lambda+1)(n-1)} > 0$, $A^{(n-1)^2} > 0$, $A^p > 0$.

(b) Sp A = Sp $A^2 = \cdots =$ Sp $A^{n-2} = 0$. Dann Cayley, da $c_1 = \cdots = c_{n-2} = 0$, $A^n = aE + bA$; hierin $a, b > 0$, da sonst imprimitiv (Bw!) Dann

$$\begin{aligned} A^{n(n-2)+2} &\sim A^2(E+A)^{n-2} \\ &\sim A^2 + A^3 + \cdots + A^{n-1} + A^n \\ &\sim E + A + A^2 + \cdots + A^{n-1} > 0. \end{aligned}$$

[1] Acknowledged with thanks to Carl de Boor.

Für

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

gilt $A^n = E + A$ und daher $A^{p-1} \not> 0$, da $A^{p-1} = A(E+A)^{n-2} \sim A + A^2 + \cdots + A^{n-1} a_{11}^{(p-1)} = 0$ hat.

Translation of Wielandt's proof[2]

Nonnegative matrices.

If A is irreducible, $A \geqslant 0$, and if the maximal root r is the only one of maximal absolute value, then for sufficiently large p, $A^p > 0$. Frobenius shows 1912: If A is irreducible, $a_{11} > 0$, then in A^{n-1} the first row and column are positive and hence $A^{2n-2} > 0$. It follows that, if A is irreducible and r is the only root of maximal absolute value (i.e., A "primitive") since also $\widehat{rE - A} > 0$, also $A\widehat{rE - A} > 0$, and is a linear combination of $A, \ldots, A^n$. Frobenius concludes that at the latest we have $a_{11}^{(m)} > 0$ for $m = n$. He does not combine both results; one would obtain $A^{n(2n-2)} > 0$.

Theorem. *The exact exponent is* $n^2 - 2n + 2 = p$. *If A is primitive, then* $A^p > 0$.

Proof. (1) If, for every $\alpha = 1, \ldots, n$, there is an l, $1 \leqslant l \leqslant n-1$, with $a_{\alpha\alpha}^l > 0$, then $A^{l(n-1)}$ has its αth column positive, thus also $A^{(n-1)^2}$, thus also $A^p > 0$.

(2) For some α, say $\alpha = 1$, let $a_{11} = a_{11}^2 = \cdots = a_{11}^{n-1} = 0$. Then $a_{11}^n \neq 0$ (e.g., by Cayley), i.e., there exist n factors $a_{1\rho} a_{\rho\sigma} a_{\sigma\tau} \cdots a_{\omega 1} \neq 0$, but not fewer. For the last reason, the indices $1, \rho, \sigma, \tau, \ldots, \omega$ are all different, thus for an appropriate ordering, $1, 2, \ldots, n$:

$$A \gtrsim \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

It follows from this, if some power A^q has a positive column, that one of the αth column is positive once among $A^q, A^{q+1}, \ldots, A^{q+(n-1)}$, and hence *every* column of $A^{q+(n+1)}$ is positive. Now two cases:

[2] With very slight emendations.

Case (a). First suppose that $\mathrm{tr}(A^{\lambda}) \neq 0$ for some $\lambda,\ 1 \leqslant \lambda \leqslant n-2$.

Then $a^{\lambda}_{\alpha\alpha} > 0$ for some α; thus $A^{\lambda(n-1)}$ has a positive αth column; hence $A^{\lambda(n-1)+(n-1)} > 0,\ A^{(\lambda+1)(n-1)} > 0,\ A^{(n-1)^2} > 0,\ A^{p} > 0$.

Case (b). $\mathrm{tr}(A) = \mathrm{tr}(A^2) = \cdots = \mathrm{tr}(A^{(n-2)}) = 0$.

Then by Cayley, since $c_1 = \cdots = c_{n-2} = 0,\ A^n = aA + bE$, with $a, b > 0$, else A (would be) imprimitive (Pf!). Thus

$$\begin{aligned} A^{n(n-2)+2} &\sim A^2(E+A)^{n-2} \\ &\sim A^2 + A^3 + \cdots + A^{n-1} + A^n \\ &\sim E + A + A^2 + \cdots + A^{n-1} > 0. \end{aligned}$$

For

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 \end{pmatrix}$$

we have $A^n = E + A$ and therefore $A^{p-1} \not> 0$ since

$$A^{p-1} = A(E+A)^{n-2} \sim A + A^2 + \cdots + A^{n-1}$$

yields $a^{p-1}_{11} = 0$. □

Commentary on Wielandt's proof

Many years ago Helmut Wielandt told me that it was his habit to make notes of his mathematical thoughts and that these were collected in notebooks he called "Tagebücher" (diaries). After he died, Heinrich Wefelscheid and I travelled to Wielandt's retirement home in Schliersee, Bavaria, and, with the help of Wielandt's wife and son, collected the diaries. The facsimile of the page (plus a few lines) of the diaries and its translation are the first of the contents to be published. They contain a proof of a result announced without proof in Wielandt's seminal [3], a paper that I have discussed in some detail in a commentary following it in Wielandt's Mathematical Works. I believe that his proof has not been published anywhere and it is likely that no one except Wielandt knew what it was.

Wielandt's note to himself begins by repeating Frobenius' definition of a *primitive* matrix in [2], that is a nonnegative irreducible matrix whose spectral radius is the only eigenvalue of maximal absolute value, and proceeds by quoting two remarks from [2]. Linked by the observation that every power of a primitive matrix is irreducible (which is omitted here) they may be combined to yield an upper bound for *the exponent of primitivity* of a primitive matrix, that is the first power that is positive. Wielandt then proceeds to prove a sharp bound for this exponent.

However it should be noted that though the proof of the second of these remarks in [2] uses much of the machinery of Perron–Frobenius, Wielandt indicates an independent proof for it by the parenthetical remark "(e.g., by Cayley)", by which he presumably means the Cayley–Hamilton Theorem. Indeed, a second proof follows easily from the positivity of $(I + A)^{n-1}$ when A is irreducible, which is proved in [3]. It therefore appears that Wielandt has nowhere used Frobenius' definition of primitivity and has given a direct proof of the following result:

If A is a nonnegative matrix and some power of A is positive, then $A^q > 0$, where $q = n^2 - 2n + 2$. Further, there is such a matrix A for which $A^{(q-1)}$ is not positive.

The only tool used by Wielandt from the theory of matrices is the Cayley–Hamilton theorem which is quoted twice without further explanation of its relevance. But it is not hard to interpolate a few steps in this and a few other places as needed. His other arguments are elementary, but highly ingenious, calculations with nonnegative matrices, and arguments with products of elements of a matrix that probably would now be presented in terms of the digraph of a nonnegative matrix. For a neat modern graph theoretic proof which is entirely self-contained (i.e., uses only elementary results from graph theory, matrix theory and number theory) see [1, Section 3.5]. It is also shown there that, up to permutation similarity, Wielandt's example near the end of his note is the only one whose exponent equals $n^2 - 2n + 2$. References will also be found in this section of [1] to proofs of Wielandt's exponent theorem going back to the late 1950's.

The page of Wielandt's diaries that is published here follows notes for [3] and contains a faint pencilled note (in German) "Paper submitted to M.Z., 2 November 1949", which is the same as the submission date in the journal. There is also a somewhat mysterious notation: "Probably end of September 1949 to Cologne (20 September)". One guess is that Wielandt may have found this result travelling to Cologne from Mainz where he was then employed. But a puzzle remains: Why did Wielandt not publish the proof of his result which has led to much subsequent work?

References

[1] R.A. Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge, 1991.
[2] F.G. Frobenius, Über Matrizen aus nicht negativen Elementen, Sitzungsber. K. Preuss. Akad. Wiss. Berlin (1912) 456–477; Ges. Abh. 3 (1968) 546–567.
[3] H. Wielandt, Unzerlegbare, nicht negative Matrizen, Math. Z. 52 (1950) 642–648 and Mathematische Werke/Mathematical Works, vol. 2, 100–106 de Gruyter, Berlin, 1996.