Basics of the Theory of Large Deviations

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Abstract

In this note, we collect basic results of the theory of large deviations.
Missing proofs can be found in the monograph [1].

1 Introduction

We start by recalling the following computation which was done in the course Wahrscheinlichkeitstheorie I (and which is also done in the course Versicherungs-

mathematik).

Assume that $X$, $X_1$, $X_2$, ... are i.i.d real-valued random variables on a proba-
bility space $(\Omega, \mathcal{F}, \mathbb{P})$ and $x \in \mathbb{R}$, $\lambda > 0$. Then, by Markov’s inequality,

$$
P \left( \sum_{i=1}^{n} X_i \geq nx \right) \leq \exp \left\{ -\lambda nx \right\} \left( \mathbb{E}(e^{\lambda X_1}) \right)^n = \exp \left\{ -n(\lambda x - \Lambda(\lambda)) \right\},$$

where $\Lambda(\lambda) := \log \mathbb{E} \exp \{ \lambda X \}$.

Defining $I(x) := \sup_{\lambda \geq 0} \{ \lambda x - \Lambda(\lambda) \}$, we therefore get

$$
P \left( \sum_{i=1}^{n} X_i \geq nx \right) \leq \exp \left\{ -nI(x) \right\},$$

which often turns out to be a good bound.
2 Cramer’s theorem for real-valued random variables

Definition 2.1. a) Let $X$ be a real-valued random variable. The function $\Lambda : \mathbb{R} \to (-\infty, \infty]$ defined by

$$\Lambda(\lambda) := \log \mathbb{E} e^{\lambda X}$$

is called cumulant generating function or logarithmic moment generating function. Let $D_\Lambda := \{\lambda : \Lambda(\lambda) < \infty\}$.

b) $\Lambda^* : \mathbb{R} \to [0, \infty]$ defined by

$$\Lambda^*(x) := \sup_{\lambda \in \mathbb{R}} \{\lambda x - \Lambda(\lambda)\}$$

is called Fenchel-Legendre transform of $\Lambda$. Let $D_{\Lambda^*} := \{\lambda : \Lambda^*(\lambda) < \infty\}$.

In the following we will often use the convenient abbreviation

$$\Lambda^*(F) := \inf_{x \in F} \Lambda^*(x)$$

for a subset $F \subseteq \mathbb{R}$ (with the usual convention that the infimum of the empty set is $+\infty$).

Lemma 2.2. a) $\Lambda$ is convex.

b) $\Lambda^*$ is convex.

c) $\Lambda^*$ is lower semi-continuous, i.e. $\{x \in E : \Lambda^*(x) \leq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$.

d) If $D_\Lambda = \{0\}$, then we have $\Lambda^* \equiv 0$.

e) If $\Lambda(\lambda) < \infty$ for some $\lambda > 0$, then we have $\mathbb{E}X < \infty$ (but possibly $\mathbb{E}X = -\infty$).

f) If $\mathbb{E}X < \infty$ (but possibly $\mathbb{E}X = -\infty$), then we have

$$\Lambda^*(x) := \sup_{\lambda \geq 0} \{\lambda x - \Lambda(\lambda)\}, \ x \geq \mathbb{E}X$$

and $\Lambda^*$ is nondecreasing on $[\mathbb{E}X, \infty)$.
g) $\mathbb{E}|X| < \infty$ implies $\Lambda^*(\mathbb{E}X) = 0$.

h) $\inf_{x \in \mathbb{R}} \Lambda^*(x) = 0$.

i) $\Lambda$ is differentiable in the interior of $D_\Lambda$ with derivative

$$\Lambda'(\eta) = \frac{1}{\mathbb{E} \exp(\eta X) \mathbb{E}(X e^{\eta X})}.$$ 

Further, $\Lambda'(\eta) = y$ implies $\Lambda^*(y) = \eta y - \Lambda(\eta)$.

Proof. [1] \hfill \Box

Examples 2.3. a) $\mathcal{L}(X) = \text{Poisson}(\theta), \theta > 0$. Then

$$\Lambda^*(x) = -x + \theta + x \log \left(\frac{x}{\theta}\right), \quad x \geq 0.$$ 

b) $\mathcal{L}(X) = \text{Bernoulli}(p), 0 < p < 1$. Then

$$\Lambda^*(x) = x \log \frac{x}{p} + (1 - x) \log \frac{1 - x}{1 - p}, \quad 0 \leq x \leq 1.$$ 

c) $\mathcal{L}(X) = \text{Exp}(\theta), \theta > 0$. Then

$$\Lambda^*(x) = \theta x - 1 - \log(\theta x), \quad x \geq 0.$$ 

d) $\mathcal{L}(X) = \mathcal{N}(0, \sigma^2), \sigma > 0$. Then

$$\Lambda^*(x) = \frac{x^2}{2\sigma^2}, \quad x \in \mathbb{R}.$$ 

In all cases $\Lambda^*(x)$ is $\infty$ for all other values of $x$.

Now we are ready to formulate and prove Cramér’s Theorem.

**Theorem 2.4.** Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed real-valued random variables and let $\mu_n := \mathcal{L}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right), n \in \mathbb{N}$. Then the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ satisfies the following properties.

a) $\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(F) \leq -\Lambda^*(F)$ for every closed set $F \subseteq \mathbb{R}$.

b) $\liminf_{n \to \infty} \frac{1}{n} \log \mu_n(G) \geq -\Lambda^*(G)$ for every open set $G \subseteq \mathbb{R}$.
For a closed set $F \subseteq \mathbb{R}$ we even have $\mu_n(F) \leq 2 \exp\{-n\Lambda^*(F)\}$ for every $n \in \mathbb{N}$ and for a closed interval $F$ of $\mathbb{R}$ we even have $\mu_n(F) \leq \exp\{-n\Lambda^*(F)\}$ for every $n \in \mathbb{N}$.

**Proof.** Obviously c) implies a), so it suffices to prove c) and b). We always assume that $X$ is a random variable with law $\mu := \mu_1$.

c) The assertions are clearly true in case $F = \emptyset$, so we assume that $F$ is closed and nonempty. The assertions are also clear in case $\Lambda^*(F) = \inf_{x \in F} \Lambda^*(x) = 0$, so we assume $\Lambda^*(F) > 0$. It follows from Lemma 2.2d) that there exists some $\lambda \neq 0$ such that $\Lambda(\lambda) < \infty$. Assume that $\lambda > 0$ (the case $\lambda < 0$ is treated analogously). Lemma 2.2e) shows that $\mathbb{E}X < \infty$. For $\lambda \geq 0$ and $x \in \mathbb{R}$ we get:

$$
\mu_n([x, \infty)) = \mathbb{P}\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i \geq x \right\} = \mathbb{P}\left\{ \exp\left( \lambda \sum_{i=1}^{n} X_i \right) \geq \exp(\lambda nx) \right\} \leq \exp(-\lambda nx) \mathbb{E}\left( \exp\left( \lambda \sum_{i=1}^{n} X_i \right) \right) = e^{-n(\lambda x - \Lambda(\lambda))}.
$$

Since $\mathbb{E}X < \infty$, Lemma 2.2f) shows that for $x \geq \mathbb{E}X$ we have

$$
\mu_n([x, \infty)) \leq e^{-n\Lambda^*(x)}. \quad (1)
$$

**Case 1:** $\mathbb{E}|X| < \infty$ (i.e. $\mathbb{E}X > -\infty$). Since $\Lambda^*(F) > 0$, Lemma 2.2g) implies $\mathbb{E}X \in F^c$. Let $(x_-, x_+)$ be the largest interval in $F^c$ which contains $\mathbb{E}X$. Since $F$ is nonempty, at least one of the numbers $x_-, x_+$ is finite. If $x_+$ is finite, then $x_+ \in F$ and

$$
\mu_n(F \cap [x_+, \infty)) \leq \mu_n([x_+, \infty)) \leq \exp\{-n\Lambda^*(x_+)\} \leq \exp\{-n\Lambda^*(F)\}.
$$

If $x_- > -\infty$, then $x_- \in F$ and

$$
\mu_n(F \cap (-\infty, x_-]) \leq \mu_n(-\infty, x_-]) \leq \exp\{-n\Lambda^*(x_-)\} \leq \exp\{-n\Lambda^*(F)\}.
$$

Hence $\mu_n(F) \leq 2 \exp\{-n\Lambda^*(F)\}$. In case $F$ is an interval either $x_- = -\infty$ or $x_+ = \infty$. 

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**Case 2:** $\mathbb{E}X = -\infty$. Lemma 2.2f) shows that the function $x \mapsto \Lambda^*(x)$ is nondecreasing and Lemma 2.2h) implies that $\lim_{x \to -\infty} \Lambda^*(x) = 0$. Since $\Lambda^*(F) > 0$ and $F \neq \emptyset$, there exists some $x_+ \in \mathbb{R}$ such that $F \subseteq [x_+, \infty)$ and $x_+ \in F$. Now (1) implies

$$
\mu_n(F) \leq \mu_n([x_+, \infty)) \leq \exp\{-n\Lambda^*(x_+)) \leq \exp\{-n\Lambda^*(F)\}.
$$

This proves part c).

**b)** Below we will show, that for every $\delta > 0$ (and every law $\mu$) we have

$$
\lim \inf_{n \to \infty} \frac{1}{n} \log \mu_n(\left(-\delta, \delta\right)) \geq \inf_{\lambda \in \mathbb{R}} \Lambda(\lambda) = -\Lambda^*(0). \tag{2}
$$

Assume this holds and let $x \in \mathbb{R}$ and $Y := X - x$. Then we have

$$
\Lambda_Y(\lambda) = \log \mathbb{E} e^{\lambda Y} = -\lambda x + \Lambda(\lambda)
$$

and

$$
\Lambda^*_Y(z) = \sup \{\lambda z - \Lambda_Y(\lambda)\} = \sup \{\lambda z + \lambda x - \Lambda(\lambda)\} = \Lambda^*(z + x).
$$

Using (2), this implies that for any $x \in \mathbb{R}$ and $\delta > 0$, we have

$$
\lim \inf_{n \to \infty} \frac{1}{n} \log \mu_n((x - \delta, x + \delta)) = \lim \inf_{n \to \infty} \frac{1}{n} \log \mu_n^{(Y)}((-\delta, \delta)) \\
\geq -\Lambda^*_Y(0) = -\Lambda^*(x).
$$

If $G = \emptyset$, then assertion b) is obvious. So assume that $G$ is open and nonempty and $x \in G$. Then we have $(x - \delta, x + \delta) \subseteq G$ for some $\delta > 0$ and hence

$$
\lim \inf_{n \to \infty} \frac{1}{n} \log \mu_n(G) \geq \lim \inf_{n \to \infty} \frac{1}{n} \log \mu_n((\left(x - \delta, x + \delta\right)) \geq -\Lambda^*(x),
$$

and – since $x \in G$ was arbitrary – we have

$$
\lim \inf_{n \to \infty} \frac{1}{n} \log \mu_n(G) \geq -\inf_{x \in G} \Lambda^*(x).
$$

It remains to show (2).

**Case 1:** $\mu((-\infty, 0)) > 0$, $\mu((0, \infty)) > 0$ and $\mu$ has compact support.
Since $\mu$ has compact support, there exists some $a > 0$ such that $\mu([-a, a]) = 1$. Further, $\Lambda(\lambda) \leq |\lambda|a < \infty$ for every $\lambda \in \mathbb{R}$. For the rest of the proof, see [1].

**Case 2:** $\mu((-\infty, 0)) > 0$, $\mu((0, \infty)) > 0$ and $\mu$ has unbounded support. Let $M$ be so large that both $\mu([-M, 0))$ and $\mu((0, M])$ are strictly positive. Below we will let $M$ tend to infinity. Define the probability measure $\nu$ on the Borel sets of $\mathbb{R}$ by

$$\nu(A) := \frac{\mu(A \cap [-M, M])}{\mu([-M, M])}.$$ 

Clearly $\nu$ satisfies the assumptions of Case 1. Defining $\nu_n$ in analogy to $\mu_n$ and letting $\Lambda_M$ denote the logarithmic moment generating function associated to $\nu$ and defining

$$\Lambda^{(M)}(\lambda) := \log \int_{-M}^{M} e^{\lambda y} d\mu(y),$$

we get

$$\mu_n((-\delta, \delta)) \geq \nu_n((-\delta, \delta)) \mu([-M, M])^n$$

and

$$\liminf \frac{1}{n} \log \mu_n((-\delta, \delta)) \geq \liminf \frac{1}{n} \log \nu_n((-\delta, \delta)) + \log \mu([-M, M]) \geq \Lambda_M(\lambda) + \log \mu([-M, M]) = \Lambda^{(M)}(\lambda).$$

It suffices to prove

$$I^* := \lim_{M \to \infty} \inf_{\lambda \in \mathbb{R}} \Lambda^{(M)}(\lambda) \geq \inf_{\lambda \in \mathbb{R}} \Lambda(\lambda). \tag{3}$$

Since $M \mapsto \inf_{\lambda \in \mathbb{R}} \Lambda^{(M)}(\lambda)$ is nondecreasing, the sets

$$\{ \lambda : \Lambda^{(M)}(\lambda) \leq I^* \}$$

are nonempty, compact and decreasing in $M$, so the intersection of all these sets is nonempty. If $\lambda_0$ is in the intersection, then

$$\Lambda(\lambda_0) = \lim_{M \to \infty} \Lambda^{(M)}(\lambda_0) \leq I^*.$$

This finishes Case 2.
Case 3: Either \( \mu((-\infty,0)) = 0 \) or \( \mu((0,\infty)) = 0 \). In this case, \( \lambda \mapsto \Lambda(\lambda) \) is either nonincreasing or nondecreasing and \( \inf_{\lambda \in \mathbb{R}} \Lambda(\lambda) = \log \mu(\{0\}) \).

Therefore

\[
\mu_n((-\delta,\delta)) \geq \mu_n(\{0\}) = (\mu(\{0\}))^n,
\]

and hence

\[
\frac{1}{n} \log \mu_n((-\delta,\delta)) \geq \log \mu(\{0\}) = \inf_{\lambda \in \mathbb{R}} \Lambda(\lambda).
\]

This proves (2) and hence Cramer’s theorem.

\[\square\]

3 Basic concepts of the theory of large deviations

In the following \( E \) denotes a topological space and \( \mathcal{E} \) the Borel sets of \( E \).

Definition 3.1. \( I : E \to [0,\infty] \) is called a rate function, in case \( I \) is lower semi-continuous (i. e. \( \{x \in E : I(x) \leq \alpha\} \) is closed for every \( \alpha \geq 0 \)). \( I \) is called a good rate function if – in addition – \( \{x \in E : I(x) \leq \alpha\} \) is compact for every \( \alpha \geq 0 \).

Again we will use the abbreviation \( I(G) := \inf\{I(x); x \in G\} \) for any subset \( G \) of \( E \).

Definition 3.2. Let \( \{\mu_n\}_{n \in \mathbb{N}} \) be a family of probability measures on \((E,\mathcal{E})\) and let \( I \) be a rate function. \( \{\mu_n\}_{n \in \mathbb{N}} \) is said to satisfy a large deviation principle (LDP) with rate function \( I \), if

a) \( \lim \inf_{n \to \infty} \frac{1}{n} \log \mu_n(G) \geq -I(G) \) for every open set \( G \)

b) \( \lim \sup_{n \to \infty} \frac{1}{n} \log \mu_n(F) \leq -I(F) \) for every closed set \( F \).

\( \{\mu_n\}_{n \in \mathbb{N}} \) is said to satisfy a weak large deviation principle with rate function \( I \), if a) holds and b) holds with “closed” replaced by “compact”.

Remark 3.3. Cramér’s Theorem says that the sequence of the laws \( \mu_n \) of the average of \( n \) i. i. d. random variables satisfies an LDP with rate function \( \Lambda^* \). \( \Lambda^* \) may or may not be a good rate function (depending on the law of the \( X_i \)).
Remark 3.4. If the topological space $E$ is Hausdorff, then every compact set is closed and hence a Borel set. If $E$ is not Hausdorff, then a compact set need not be Borel and that causes a problem when formulating a weak LDP. One way out is to replace “compact” by “compact and closed”. Further below, we will assume that $E$ is Hausdorff and therefore this will not be a problem for us.

Remark 3.5. Why do we require that a rate function $I$ be lower semi-continuous? Well, assume that $I$ is any function (not necessarily lower semi-continuous) from $E$ to $[\infty, \infty]$ such that $\{\mu_n\}_{n \in \mathbb{N}}$ satisfies a (weak) large deviation principle with function $I$. For $x \in E$, define

$$\tilde{I}(x) := \sup_{x \in G \text{ open}} I(G).$$

Then $\tilde{I}(x) \leq I(x)$, since $\inf_{y \in G} I(y) \leq I(x)$ for every open set $G$ containing $x$. Further, if $G$ is an open set containing $x$, then $\tilde{I}(x) \geq I(G)$ which implies $\tilde{I}(G) \geq I(G)$, so we have $\tilde{I}(G) = I(G)$ for every open set $G$ containing $x$. Therefore $\{\mu_n\}_{n \in \mathbb{N}}$ satisfies a (weak) large deviation principle with function $\tilde{I}$ as well. Furthermore $\tilde{I}$ is lower semi-continuous: fix $\alpha \in \mathbb{R}$; we show that $\{x \in E : \tilde{I}(x) > \alpha\}$ is open. Let $x \in E$ satisfy $\tilde{I}(x) = \beta > \alpha$. By definition of $\tilde{I}$ there exists an open set $G$ containing $x$ such that $I(G) > \alpha$ and therefore $\tilde{I}(G) > \alpha$ showing that $\tilde{I}$ is a rate function. $\tilde{I}$ is called the lower semi-continuous regularization of $I$.

Remark 3.6. Suppose $\{\mu_n\}_{n \in \mathbb{N}}$ satisfies an LDP with rate functions $I$ and $I'$. Is it then true that $I = I'$, i.e. is the rate function uniquely determined by $\{\mu_n\}_{n \in \mathbb{N}}$? For any “reasonable” topological space $E$ this is true. It is more than enough to assume that $E$ is a metric space. So, let us suppose that $E$ is a metric space and $\{\mu_n\}_{n \in \mathbb{N}}$ satisfies a weak LDP with rate function $I$. Fix $x \in E$. Then for any open set $G$ containing $x$ we have

$$-I(x) \leq -I(G) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(G). \quad (4)$$

On the other hand, for any $\varepsilon > 0$ we find an open set $G_1$ containing $x$ such that $I(G_1) > I(x) - \varepsilon$ since $I$ is lower semi-continuous. Since $E$ is a metric space we can find another open set $G_2$ containing $x$ such that $\bar{G}_2$ (the closure of $G_2$) is contained in $G_1$. Therefore

$$-I(x) \geq -I(G_2) - \varepsilon \geq \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(G_2) - \varepsilon$$

$$\geq \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(G_2) - \varepsilon.$$
Since $\varepsilon > 0$ was arbitrary this shows – together with (4) for $G = G_2$ – that $I(x)$ is uniquely determined by the family $\{\mu_n\}_{n \in \mathbb{N}}$. The argument goes through for topological spaces which are regular (see [1]).

The example below shows that uniqueness does not hold on every topological space.

**Example 3.7.** Let $E := \{a, b\}$ and let $\mathcal{T} = \{\emptyset, \{a\}, E\}$ be a topology on $E$, $(E, \mathcal{T})$ is called Sierpinski space. Observe that $(E, \mathcal{T})$ is not a Hausdorff space. Let $\mu_n := \delta_a$ be a unit point mass at $a$ for every $n$. Then $\{\mu_n\}_{n \in \mathbb{N}}$ satisfies a LDP with (good) rate function $I$ for any function $I : E \to [0, \infty]$ which satisfies $I(b) = 0$, i.e. $I(a)$ can be chosen arbitrarily in $[0, \infty]$.

From now on we will assume that the space $(E, \mathcal{T})$ is Hausdorff. This guarantees in particular that every compact set is closed and hence measurable.

**Theorem 3.8.** *(Contraction Principle)* Let $(E_1, \mathcal{T}_1)$ and $(E_2, \mathcal{T}_2)$ be (Hausdorff) topological spaces and let $T : E_1 \to E_2$ be continuous. Further assume that $\{\mu_n\}_{n \in \mathbb{N}}$ satisfies an LDP with good rate function $I_1$ on $(E_1, \mathcal{T}_1)$. Define $I_2 : E_2 \to [0, \infty]$ by

$$I_2(y) := \inf \{I_1(x) : x \in E_1, T(x) = y\}.$$ 

Then $\{\mu_n \circ T^{-1}\}_{n \in \mathbb{N}}$ satisfies an LDP with good rate function $I_2$.

**Proof.** We first check that $I_2$ is a good rate function. Let $\alpha \geq 0$. Then

$$\{y : I_2(y) \leq \alpha\} = T(\{x : I_1(x) \leq \alpha\});$$

“⊇” is clear and “⊆” follows since $I_1$ is good, which implies that the infimum in the definition of $I_2$ is attained whenever $I_2(y) < \infty$. Since the continuous map $T$ maps compact subsets of $E_1$ to compact subsets of $E_2$, it follows that $I_2$ is a good rate function (we don’t need to check lower semi-continuity since we assumed $E_2$ to be Hausdorff).

Observe that for any $A \subseteq E_2$, we have

$$\inf_{y \in A} I_2(y) = \inf_{x \in T^{-1}(A)} I_1(x).$$

Since $T$ is continuous, $T^{-1}(A)$ is open (resp. closed) if $A$ is open (resp. closed). Therefore the LDP for $\{\mu_n \circ T^{-1}\}_{n \in \mathbb{N}}$ follows as a consequence of the LDP for $\{\mu_n\}_{n \in \mathbb{N}}$. □
Remark 3.9. If $I_1$ is a rate function but not a good one, then $I_2$ as defined in the previous theorem need not even be a rate function. As an example, take $E_1 = E_2 = \mathbb{R}$, $I_1 \equiv 0$ and $T(x) = \exp(x)$.

Definition 3.10. $\{\mu_n\}_{n \in \mathbb{N}}$ is called \textit{exponentially tight}, if for every $0 < \alpha < \infty$ there exists a compact set $K_\alpha$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(K_\alpha^c) < -\alpha.$$ 

Remark 3.11. If $\{\mu_n\}_{n \in \mathbb{N}}$ is exponentially tight and $E$ is Polish, then $\{\mu_n\}_{n \in \mathbb{N}}$ is tight. To see this, pick $0 < \varepsilon < 1$ and define $\alpha := -\log \varepsilon$. By assumption there exists a compact set $K$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(K^c) < -\alpha = \log \varepsilon,$$

so there exists some $n_0$ such that for all $n \geq n_0$ we have

$$\mu_n(K^c) < \exp\{-\alpha n\} = \varepsilon^n < \varepsilon.$$ 

Since $E$ is Polish, a single probability measure (and hence every finite set of probability measures) is always tight and therefore the family $\{\mu_n\}_{n \in \mathbb{N}}$ is tight.

Lemma 3.12. Assume that $\{\mu_n\}_{n \in \mathbb{N}}$ is exponentially tight and that $I$ is a rate function.

a) If

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(F) \leq -I(F)$$

for every compact set $F$, then the same is true for every closed set $F$ (this is even true without the assumption that $I$ is a rate function).

b) If

$$\liminf_{n \to \infty} \frac{1}{n} \log \mu_n(G) \geq -I(G)$$

for every open set $G$, then $I$ is a good rate function.

So: If $\{\mu_n\}_{n \in \mathbb{N}}$ is exponentially tight and satisfies a weak LDP with rate function $I$, then $\{\mu_n\}_{n \in \mathbb{N}}$ even satisfies an LDP and $I$ is good.
Proof. a) Let $F$ be a closed set in $E$ and $\alpha := \inf_{x \in F} I(x)$. First assume that $\alpha < \infty$. Let $K_{\alpha}$ be as in the definition of exponential tightness. Clearly

$$F \cap K_{\alpha} \subseteq F \subseteq \{ I \geq \alpha \}$$

and

$$\mu_n(F) \leq \mu_n(F \cap K_{\alpha}) + \mu_n(K_{\alpha}^c).$$

Hence

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(F) \leq \limsup_{n \to \infty} \frac{1}{n} \log (2 \max \{ \mu_n(F \cap K_{\alpha}), \mu_n(K_{\alpha}^c) \})$$

$$= \limsup_{n \to \infty} \max \left\{ \frac{1}{n} \log \mu_n(F \cap K_{\alpha}), \frac{1}{n} \log \mu_n(K_{\alpha}^c) \right\}$$

$$\leq -\alpha = -\inf_{x \in F} I(x).$$

If $\alpha = \infty$, then replace $\alpha$ by $M$ in the above arguments and then let $M \to \infty$.

b) Fix $\alpha \geq 0$. Let $K_{\alpha}$ be as in the definition of exponential tightness. Then $K_{\alpha}^c$ is open and

$$-\inf_{x \in K_{\alpha}^c} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(K_{\alpha}^c) < -\alpha$$

and therefore $\{ I \leq \alpha \} \subseteq K_{\alpha}$. Since $\{ I \leq \alpha \}$ is closed by assumption ($I$ is a rate function) and every closed subset of a compact set is compact (this holds on any topological space), we see that $I$ is a good rate function.

The following proposition is a partial converse of the previous lemma.

**Proposition 3.13.** Let $E$ be a locally compact Hausdorff space and assume that $\{ \mu_n \}_{n \in \mathbb{N}}$ satisfies an LDP with good rate function $I$. Then $\{ \mu_n \}_{n \in \mathbb{N}}$ is exponentially tight.

**Proof.** Fix $\alpha \geq 0$. Then the set $\{ x \in E : I(x) \leq \alpha \}$ is compact (since $I$ is good). By local compactness every $x \in E$ has a compact neighborhood. We cover $\{ x \in E : I(x) \leq \alpha \}$ by the family of interiors of all these compact neighborhoods for all $x \in \{ x \in E : I(x) \leq \alpha \}$. By compactness there exists a finite subcover and the union of the corresponding compact neighborhoods is itself compact (finite
unions of compact sets are always compact). Let $K$ denote this compact set. Then
\[ \{ x \in E : I(x) \leq \alpha \} \subset \text{int}(K) \subset K \]
and hence
\[ K^c \subset \text{cl}(K^c) \subset \{ x \in E : I(x) > \alpha \}, \]
where \( \text{int}(A) \) and \( \text{cl}(A) \) denote the interior and the closure of a set \( A \). We get
\[ \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(K^c) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(\text{cl}(K^c)) \leq -\inf_{x \in \text{cl}(K^c)} I(x) \leq -\alpha. \]

\[ \square \]

4 Sanov’s theorem

Our goal in this section is to prove Sanov’s Theorem, which states an LDP for the empirical distribution of a sequence of i. i. d. random variables. The rate function turns out to be the well-known relative entropy (with respect to the joint law of the random variables). In this section we always assume that \( E \) is a Polish space with complete metric \( \rho \). We denote the set of all probability measures on \( (E, \mathcal{E}) \) by \( \mathcal{M}_1(E) \). We will need a topology on \( \mathcal{M}_1(E) \).

**Proposition 4.1.** Define \( d : \mathcal{M}_1(E) \times \mathcal{M}_1(E) \to \mathbb{R} \) by
\[ d(\mu, \nu) := \inf \{ \delta > 0 : \mu(F) \leq \nu(F^\delta) + \delta \text{ for all closed sets } F \subseteq E \}, \]
where \( F^\delta := \{ x \in E : \rho(x, F) < \delta \} \). Then \( d \) is a metric (the Lévy metric) on \( \mathcal{M}_1(E) \). With this metric \( \mathcal{M}_1(E) \) is a Polish space. Furthermore convergence with respect to this metric is the same as weak convergence.

In the following, \( X_1, X_2, \ldots \) will denote an \( E \)-valued sequence of i. i. d. random variables with law \( \mu \).

**Proposition 4.2.**
\[ \mu_n(\omega) := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(\omega)} \to \mu \quad \text{almost surely.} \]
Proof. Let \( f : E \to \mathbb{R} \) be a bounded measurable function. By the strong law of large numbers, we have
\[
\int f(x) \mu_n(\omega, \, dx) = \frac{1}{n} \sum_{i=1}^{n} f(X_i(\omega)) \to \mathbb{E}f(X_1) = \int f(x) \mu(dx) \text{ a.s.} \tag{5}
\]
In particular this holds for any bounded and continuous function. Unfortunately we cannot deduce immediately that \( \mu_n(\omega, \cdot) \) converges to \( \mu \) weakly almost surely because the exceptional sets of measure zero depend on the function \( f \). Therefore we will show the following:
\[
P\left( \lim \inf_{n \to \infty} \mu_n(\omega, G) \geq \mu(G) \text{ for all open sets } G \right) = 1, \tag{6}
\]
from which the assertion follows by the Portmanteau Theorem (WT II). Let \( G_1, G_2, \ldots \) be a countable base of the topology of \( E \), i.e. a countable family of open sets such that any open set is the union of a subfamily of the \( G_i \). We can and will assume that the family is closed with respect to taking finite unions. By (5) we have
\[
P\left( \lim_{n \to \infty} \mu_n(\omega, G_i) = \mu(G_i) \text{ for all } i \in \mathbb{N} \right) = 1.
\]
Let \( A := \{ \omega : \mu_n(\omega, G_i) \to \mu(G_i) \text{ for all } i \in \mathbb{N} \} \). Then \( P(A) = 1 \). Any open set \( G \) can be written as a union \( G = \bigcup_{k=1}^{\infty} G_{i_k} \). Then for any \( \omega \in A \) we have
\[
\mu_n(\omega, G) = \mu_n(\omega, \bigcup_{k=1}^{\infty} G_{i_k}) \geq \mu_n(\omega, \bigcup_{k=1}^{N} G_{i_k}) \to \mu(\bigcup_{k=1}^{\infty} G_{i_k})
\]
and hence
\[
\lim \inf_{n \to \infty} \mu_n(\omega, G) = \mu(G).
\]
This shows (6) and the proposition is proved. \( \square \)

Let \( \mu_n \) be defined as in the previous proposition and denote the law of the \( \mathcal{M}_1(E) \)-valued random variable \( \mu_n \) by \( L_n \) (\( L_n \) is a probability measure on \( \mathcal{M}_1(E) \)).

**Lemma 4.3.** \( \{ L_n \}_{n \in \mathbb{N}} \) is exponentially tight.

**Proof.** \( \{ \mu \} \) is tight, since \( E \) is Polish. Hence there exist compact subsets \( \Gamma_l \subseteq E \) \( l \in \mathbb{N} \) such that \( \mu(\Gamma_l^c) \leq e^{-2l^2(l^3 - 1)} \). Now define
\[
K^l := \{ \nu : \nu(\Gamma_l) \geq 1 - \frac{1}{l} \} \quad \text{and} \quad K_L := \cap_{l=1}^{\infty} K_l, \quad L = 2, 3, \ldots
\]

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Now it is not hard to see, that the set $K_L$ is tight and closed and therefore – by Prochorov’s Theorem – compact and that
\[ \limsup_{n \to \infty} \frac{1}{n} \log L_n (K_L^c) \leq -L \]
showing that $\{L_n\}_{n \in \mathbb{N}}$ is exponentially tight. For details, see [1].

**Definition 4.4.** Let $\mu, \nu \in \mathcal{M}_1(E)$. Then
\[ H(\nu|\mu) := \left\{ \begin{array}{ll} \int_E f \log f \, d\mu & \text{if } \nu \ll \mu \text{ and } f = \frac{d\nu}{d\mu}, \\ \infty & \text{otherwise.} \end{array} \right. \]
is called relative entropy (or Kullback-Leibler distance) of $\nu$ relative to $\mu$.

**Remark 4.5.**
- $H(\mu|\mu) = 0$ for every $\mu \in \mathcal{M}_1(E)$.
- In general $H(\nu|\mu) \neq H(\mu|\nu)$.
- $H(\nu|\mu) \geq 0$, since – in case $\nu \ll \mu$ –
\[ H(\nu|\mu) = \int \left( \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} \right) \, d\mu \geq \int \frac{d\nu}{d\mu} \log \int \frac{d\nu}{d\mu} \, d\mu = 0, \]
by Jensen’s inequality applied to the convex function $x \mapsto x \log x$ on $[0, \infty)$.

**Lemma 4.6.**
\[ H(\nu|\mu) = \sup \left( \int_E f \, d\nu - \log \int_E e^f \, d\mu \right) \]
\[ = \sup_{f \in C_b(E)} \left( \int_E f \, d\nu - \log \int_E e^f \, d\mu \right), \]
where the first supremum is taken over all measurable functions $f : E \to \mathbb{R}$ such that $\int_E e^f \, d\mu < \infty$ and $\int_E f \, d\nu$ is defined and where $C_b(E)$ denotes the set of bounded continuous real-valued functions on $E$.

**Proof.** We will see in the proof, that the first equality is even true in an arbitrary measurable space $(E, \mathcal{E})$. 

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Step 1: We prove “≥” in the first inequality. There is nothing to prove in case ν is not absolutely continuous with respect to µ, so we assume ν ≪ µ. Let f : E → ℝ be measurable such that ∫ e^f dµ < ∞ and ∫ f dν > −∞ (there is nothing to prove if ∫ f dν = −∞). Since ∫ e^f dµ > 0, the formula 

\[ d\mu_f(x) := \frac{e^{f(x)}}{\int e^{f} d\mu} d\mu(x) \]

defines a probability measure which is equivalent to µ (i.e. µ_f ≪ µ and µ ≪ µ_f).

Since ν ≪ µ we have

\[ H(ν|µ) = \int f dν - \log \int e^{f} dµ = H(ν|µ_f) + \int_E f dν - \log \int_E e^{f} dµ \]

where we have used that \( \frac{dν}{dµ} = \frac{dν}{dµ_f} \frac{dm_f}{dµ} \) µ- and hence ν-almost surely (WT II, 1.46).

Step 2: We prove “≤” in the first inequality. In case ν ≪ µ we define \( f := \log \frac{dν}{dµ} \).

Then

\[ H(ν|µ) = \int_E f dν - \log \int_E e^{f} dµ \]

so we have equality in this case. It remains to show “≤” in case ν is not absolutely continuous with respect to µ. Take a set \( A ∈ E \) such that ν(A) > 0, but µ(A) = 0. Define \( f_M(x) := M1_A(x) \), \( M ∈ ℕ \). Then \( H(ν|µ) = \int_E f dν - \log \int_E e^{f} dµ = Mν(A) → ∞ \) as \( M → ∞ \). This completes Step 2.

Step 3: We prove the second inequality. Clearly we have “≥” (we take the supremum over a smaller set of functions on the right hand side of the equality), so it only remains to show “≤” (see [1]).

\[ \text{Corollary 4.7. For every } µ ∈ \mathcal{M}_1(E), \text{ } H(.|µ) \text{ is a rate function.} \]
Proof. We already showed that \( H(\cdot|\mu) \) is nonnegative. To show that \( H(\cdot|\mu) \) is lower semi-continuous, fix \( \alpha \geq 0 \). Then

\[
\{ \nu : H(\nu|\mu) \leq \alpha \} = \bigcap_{f \in C_b(E)} \left\{ \nu : \int f d\nu \leq \log \int_E e^f d\mu + \alpha \right\}
\]

is an intersection of closed sets and hence closed. \( \square \)

Now we formulate Sanov’s Theorem.

**Theorem 4.8. (Sanov)** Let \( X_1, X_2, \ldots \) be an \( E \)-valued sequence of i. i. d. random variables with law \( \mu \). Define \( \mu_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i} \), \( n \in \mathbb{N} \) and \( L_n := L(\mu_n) \). Then the sequence \( \{L_n\}_{n \in \mathbb{N}} \) satisfies an LDP with good rate function \( H(\cdot|\mu) \).

Proof. We know from the previous corollary that \( H(\cdot|\mu) \) is a rate function and from Lemma 4.3 that \( \{L_n\}_{n \in \mathbb{N}} \) is exponentially tight. It remains to verify properties a) and b) in Lemma 3.12 for \( \{L_n\}_{n \in \mathbb{N}} \) (see [1]). \( \square \)

**References**
