On the Average Complexity of the k-Level*

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Abstract

Let $\mathcal{L}$ be an arrangement of n lines in the Euclidean plane. The *k-level* of $\mathcal{L}$ consists of all vertices v of the arrangement which have exactly k lines of $\mathcal{L}$ passing below v. The complexity (the maximum size) of the k-level in a line arrangement has been widely studied. In 1998 Dey proved an upper bound of $O(n\cdot(k+1)^{1/3})$. Due to the correspondence between lines in the plane and great-circles on the sphere, the asymptotic bounds carry over to arrangements of great-circles on the sphere, where the k-level denotes the vertices at distance at most k to the south pole.

We prove an upper bound of $O((k+1)^2)$ on the expected complexity of the k-level in great-circle arrangements if the south pole is chosen uniformly at random among all cells.

We also consider arrangements of great $(d-1)$-spheres on the sphere $\mathbb{S}^d$ which are orthogonal to a set of random points on $\mathbb{S}^d$. In this model, we prove that the expected complexity of the k-level is of order $\Theta((k+1)^{d-1})$.

1 Introduction

Let $\mathcal{L}$ be an arrangement of n lines in the Euclidean plane. The *vertices* of $\mathcal{L}$ are the intersection points of lines of $\mathcal{L}$. Throughout this article we consider arrangements with the properties that no line is vertical and no three lines intersect in a common vertex. The *k-level* of $\mathcal{L}$ consists of all vertices v which have exactly k lines of $\mathcal{L}$ below v. We denote the k-level by $V_k(\mathcal{L})$ and its size by $f_k(\mathcal{L})$. Moreover, by $f_k(n)$ we denote the maximum of $f_k(\mathcal{L})$ over all arrangements $\mathcal{L}$ of n lines, and by $f(n) = f_{\lfloor (n-2)/2 \rfloor}(n)$ the maximum size of the *middle level*.

A *k-set* of a finite point set P in the Euclidean plane is a subset K of k elements of P that can be separated from $P \setminus K$ by a line. Paraboloid duality is a bijection $P \leftrightarrow \mathcal{L}_P$ between point sets and line arrangements (for details on this duality see [17, Chapter 6.5] or [8, Chapter 1.4]). The number of k-sets of P equals $|V_{k-1}(\mathcal{L}_P) \cup V_{n-1-k}(\mathcal{L}_P)|$.

* M.-K. Chiu was supported by ERC StG 757609. S. Felsner and M. Scheucher were supported by DFG Grant FE 340/12-1. R. Steiner was supported by DFG-GRK 2434. P. Schnider was supported by the SNSF Project 200021E-171681. P. Valtr was supported by the grant no. 18-19158S of the Czech Science Foundation (GAČR) and by the PRIMUS/17/SCI/3 project of Charles University. This work was initiated at a workshop of the collaborative DACH project *Arrangements and Drawings* in Schloss St. Martin, Graz. We thank the organizers for the inspiring atmosphere. We also thank Birgit Vogtenhuber for interesting discussions.

36th European Workshop on Computational Geometry, Würzburg, Germany, March 16–18, 2020.
This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

In discrete and computational geometry bounds on the number of k-sets of a planar point set, or equivalently on the size of k-levels of a planar line arrangement have important applications. The complexity of k-levels was first studied by Lovász [14] and Erdős et al. [11]. They bound the size of the k-level by $O(n \cdot (k+1)^{1/2})$. Dey [6] used the crossing lemma to improve the bound to $O(n \cdot (k+1)^{1/3})$. In particular, the maximum size $f(n)$ of the middle level is $O(n^{4/3})$. Concerning the lower bound on the complexity, Erdős et al. [11] gave a construction showing that $f(2n) \geq 2f(n) + cn = \Omega(n \log n)$ and conjectured that $f(n) \geq \Omega(n^{1+\varepsilon})$. An alternative $\Omega(n \log n)$-construction was given by Edelsbrunner and Welzl [10]. The current best lower bound $f_k(n) \geq n \cdot e^{\Omega(\sqrt{\log k})}$ was obtained by Nivasch [16] improving on a bound by Tóth [22].

1.1 Generalized Zone Theorem

In order to define "zones", let us introduce the notion of "distances". For x and x' being a vertex, edge, line, or cell of an arrangement $\mathcal{L}$ of lines in $\mathbb{R}^2$ we let their *distance* $\operatorname{dist}_{\mathcal{L}}(x, x')$ be the minimum number of lines of $\mathcal{L}$ intersected by the interior of a curve connecting a point of x with a point of x'. Pause to note that the k-level of $\mathcal{L}$ is precisely the set of vertices which are at distance k to the bottom cell.

The $(\leq j)$-zone $Z_{\leq j}(\ell, \mathcal{L})$ of a line ℓ in an arrangement $\mathcal{L}$ is defined as the set of vertices, edges, and cells from $\mathcal{L}$ which have distance at most j from ℓ. See Figure 1a for an illustration.

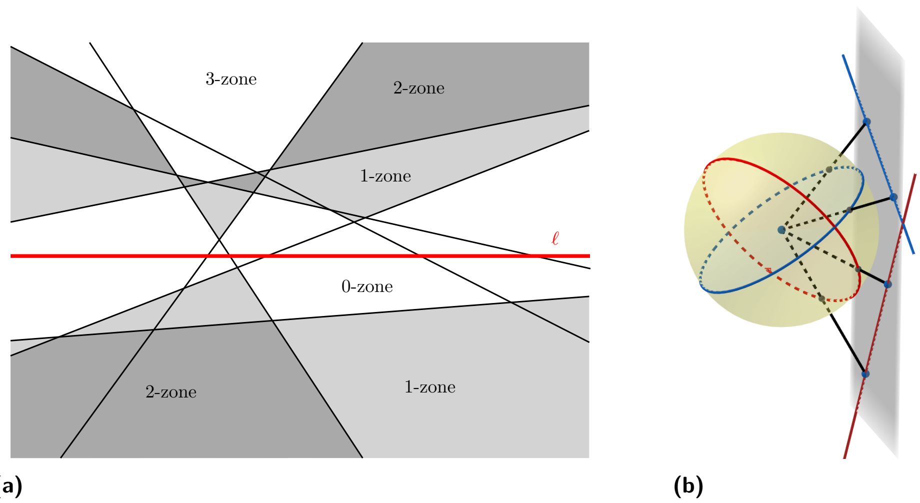

Figure 1 (a) The higher order zones of a line ℓ. (b) The correspondence between great-circles on the unit sphere and lines in a plane. Using the center of the sphere as the center of projection points on the sphere are projected to the points in the plane.

For arrangements of hyperplanes in $\mathbb{R}^d$ the $(\leq j)$-zone is defined alike. The classical zone theorem provides bounds for the zone ((≤ 0)-zone) of a hyperplane (cf. [9] and [15, Chapter 6.4]). A generalization with bounds for the complexity of the $(\leq j)$-zone appears as an exercise in Matoušek's book [15, Exercise 6.4.2]. In the proof of Theorem 2.1 we use a variant of the 2-dimensional case (Theorem 1.1). For the sake of completeness and to provide explicit constants, we include the proof in the full version [5].

▶ **Theorem 1.1.** *Let $\mathcal{L}$ be a simple arrangement of n lines in $\mathbb{R}^2$ and $\ell \in \mathcal{L}$. The $(\leq j)$-zone of ℓ contains at most $2e \cdot (j+2)n$ vertices strictly above ℓ.*

1.2 Arrangements of Great Circles

Let Π be a plane in 3-space which does not contain the origin and let $\mathbb{S}^2$ be a sphere in 3-space centered at the origin. The central projection Ψ_Π yields a bijection between arrangements of great circles on $\mathbb{S}^2$ and arrangements of lines in Π. Figure 1b gives an illustration.

The correspondence Ψ_Π preserves intersesting properties, e.g. simplicity of the arrangements. If $\Psi_\Pi(\mathcal{C}) = \mathcal{L}$, and $\mathcal{L}$ has no parallel lines, then Ψ_Π induces a bijection between pairs of antipodal vertices of $\mathcal{C}$ and vertices of $\mathcal{L}$.

As in the planar case, we define the *distance* between points x, y of $\mathbb{S}^2$ relative to a great-circle arrangement $\mathcal{C}$ as the minimum number of circles of $\mathcal{C}$ intersected by the interior of a curve connecting x with y. The *k-level* ($\leq$ *k-zone* resp.) of $\mathcal{C}$ is the set of all the vertices of $\mathcal{C}$ at distance k (distance at most k resp.) from the south pole.

Let Π_1 and Π_2 be two parallel planes in 3 space with the origin between them and let Ψ_1 and Ψ_2 be the respective central projections. For a great-circle arrangement $\mathcal{C}$ we consider $\mathcal{L}_1 = \Psi_1(\mathcal{C})$ and $\mathcal{L}_2 = \Psi_2(\mathcal{C})$. A vertex v from the k-level of $\mathcal{C}$ maps to a vertex of the k-level in one of $\mathcal{L}_1$, $\mathcal{L}_2$ and to a vertex of the $(n-k-2)$-level in the other. Hence, bounds for the maximum size of the k-level of line arrangements carry over to the k-level of great-circle arrangements except for a multiplicative factor of 2.

The $(\leq j)$-zone of a great-circle C in $\mathcal{C}$ projects to a $(\leq j)$-zone of a line in each of $\mathcal{L}_1$ and $\mathcal{L}_2$. Hence, the complexity of a $(\leq j)$-zone in $\mathcal{C}$ is upper bounded by two times the maximum complexity of a $(\leq j)$-zone in a line arrangement. Theorem 1.1 implies that the $(\leq j)$-zone of a great-circle C in an arrangement of n great-circles contains at most $4e \cdot (j+2)n$ vertices.

1.3 Higher Dimensions

The problem of determining the complexity of the k-level admits a natural extension to higher dimensions. We consider arrangements in $\mathbb{R}^d$ of hyperplanes with the properties that no hyperplane is parallel to the x_d-axis and no $d+1$ hyperplanes intersect in a common point. The *k-level* $V_k(\mathcal{A})$ of $\mathcal{A}$ consists of all vertices (i.e. intersection points of d hyperplanes) which have exactly k hyperplanes of $\mathcal{A}$ below them (with respect to the d-th coordinate). We denote the k-level by $V_k(\mathcal{A})$ and its size by $f_k(\mathcal{A})$. Moreover, by $f_k^{(d)}(n)$ we denote the maximum of $f_k(\mathcal{A})$ among all arrangements $\mathcal{A}$ of n hyperplanes in $\mathbb{R}^d$.

As in the planar case, there remains a gap between lower and upper bounds;

$$\Omega(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - 1}) \leq f_k^{(d)}(n) \leq O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - c_d}),$$

here $c_d > 0$ is a small positive constant only depending on d. Details and references can be found in Chapter 11 of Matoušek's book [15]. In dimensions 3 and 4 improved bounds have been established. For example, for $d = 3$, it is known that $f_k^{(3)}(n) \leq O(n(k+1)^{3/2})$ (see [21]). For the middle level in dimension $d \geq 2$ an improved lower bound $f^{(d)}(n) \geq n^{d-1} \cdot e^{\Omega(\sqrt{\log n})}$ is known (see [22] and [16]).

We call the intersection of $\mathbb{S}^d$ with a central hyperplane in $\mathbb{R}^{d+1}$ a *great-$(d-1)$-sphere* of $\mathbb{S}^d$. Similar to the planar case, arrangements of hyperplanes in $\mathbb{R}^d$ are in correspondence with arrangements of great-$(d-1)$-spheres on the unit sphere $\mathbb{S}^d$ (embedded in $\mathbb{R}^{d+1}$). The terms “distance” and “k-level” generalize in a natural way.

2 Our Results

In the first part of this paper we consider arrangements of great-circles on the sphere and investigate the average complexity of the k-level when the southpole is chosen uniformly at random among the cells. This question was raised by Barba, Pilz, and Schnider while sharing a pizza [4, Question 4.2].

In Section 3 we prove the following bound on the average complexity.

▶ **Theorem 2.1.** *Let $\mathcal{C}$ be a simple arrangement of n great-circles. For $k < n/3$ the expected size of the k-level is at most $4e \cdot (k+2)^2$ when the southpole is chosen uniformly at random among the cells of $\mathcal{C}$.*

The condition $k < n/3$ is needed for Lemma 3.2 as for larger k we would have to double the multiplicative constant. However, for k in $\Omega(n^{3/5})$ the stated bound is implied by the $O(nk^{1/3})$ bound on the maximum size of a k-level. Still it is remarkable that the bound is independent of the number n of great-circles in the arrangement.

In the second part, we investigate arrangements of randomly chosen great-circles. Here we propose the following model of randomness. On $\mathbb{S}^2$ we have the duality between points and great-circles (each antipodal pair of points defines the normal vector of the plane containing a great-circle). Since we can choose points uniformly at random from $\mathbb{S}^2$, we get random arrangements of great-circles. The duality generalizes to higher dimensions so that we can talk about random arrangements on $\mathbb{S}^d$ for a fixed dimension $d \geq 2$. Using the duality between antipodal pairs of points on $\mathbb{S}^d$ and great-$(d-1)$-spheres, we prove the following bound on the expected size of the k-level in this random model (the proof can be found in the full version [5]). Again the bound does not depend on the size of the arrangement.

▶ **Theorem 2.2.** *Let $d \geq 2$ be fixed. In an arrangement of n great-$(d-1)$-spheres chosen uniformly at random on the unit sphere $\mathbb{S}^d$ (embedded in $\mathbb{R}^{d+1}$), the expected size of the k-level is of order $\Theta((k+1)^{d-1})$ for all $k \leq n/2$.*

3 Proof of Theorem 2.1

For the proof of Theorem 2.1, we fix a great-circle C from $\mathcal{C}$ and denote the closures of the two hemispheres of C on $\mathbb{S}^2$ as C^+ and C^-. As an intermediate step, we bound the size of the set $\mathcal{F}_k(C^+)$ of pairs (F, v), where F is a cell of C^- touching C and v is a vertex of C^+ whose distance to F is k. We show $|\mathcal{F}_k(C^+)| \leq 2e \cdot (k+1)^2 n$. In the case $k = 0$, vertex v must be one of the $2n$ vertices on C and F is one of the two cells of C^- which is adjacent to v. Hence, we obtain $|\mathcal{F}_0(C^+)| \leq 4n$. It remains to deal with the general case $k \geq 1$. Note that if $(v, F) \in \mathcal{F}_k(C^+)$ then v belongs to the $(\leq k-1)$-zone of C.

Consider a family $\mathcal{I}$ of half-intervals in $\mathbb{R}$, it consists of *left-intervals* of the form $(-\infty, a]$ and *right-intervals* $[b, \infty)$. A subset J of k half-intervals from $\mathcal{I}$ is a *k-clique* if there is a point $p \in \mathbb{R}$ that lies in all the half-intervals of J but not in any half-interval of $\mathcal{I} \setminus J$.

▶ **Lemma 3.1.** *Any family $\mathcal{H}$ of half-intervals in $\mathbb{R}$ contains at most $k+1$ different k-cliques.*

Proof. For $p \in \mathbb{R}$, let $l(p)$ be the number of left-intervals and $r(p)$ the number of right-intervals containing p. A point p certifies a k-clique if and only if $l(p) + r(p) = k$. From the monotonicity of the functions l and r it follows that if $(l(p_1), r(p_1)) = (l(p_2), r(p_2))$ for two points p_1 and p_2, then they are contained in the same intervals. Thus the number of k-cliques is at most the number of pairs (l, r) such that $l + r = k$ and $l, r \geq 0$, which is $k+1$. ◀

The next lemma is a corresponding result for half-circles on the circle $\mathbb{S}^1$.

▶ **Lemma 3.2.** *Any family $\mathcal{H}$ of n half-circles in $\mathbb{S}^1$ with $n > 3k$ contains at most $k+1$ different k-cliques.*

Proof. For this proof, we embed $\mathbb{S}^1$ as the unit-circle in $\mathbb{R}^2$, which is centered at the origin $\mathbf{o}$. We consider the set X of all points from $\mathbb{S}^1$, which are contained in precisely k of the half-circles of $\mathcal{I}$, and distinguish the following two cases.

Case 1: The origin $\mathbf{o}$ is not contained in the convex hull of X. There is a line separating $\mathbf{o}$ from X and rotational symmetry allows us to assume that X is contained in $\Pi^+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. For each half-circle $C \in \mathcal{H}$, the central projection of $C \cap \Pi^+$ to the line $y = 1$ is a half-interval. Since k-cliques of $\mathcal{H}$ and k-cliques of the half-intervals are in bijection we get from Lemma 3.1 that $\mathcal{H}$ has at most $k + 1$ different k-cliques.

Case 2: The origin $\mathbf{o}$ is contained in the convex hull of X. By Carathéodory's theorem, we can find three points p_1, p_2, p_3 such that $\mathbf{o}$ lies in the convex hull of p_1, p_2, p_3. Since each of the n half-circles from $\mathcal{H}$ contains at least one of these three points, and each of these three points lies on precisely k half-circles, we have $n \leq 3k$ – a contradiction to $n > 3k$. ◀

For a fixed vertex v in the $(\leq k-1)$-zone of C with $v \in C^+$, let $\mathcal{B}_{C^+}(v)$ be the set of cells F such that $(F, v) \in \mathcal{F}_k(C^+)$, in particular $\mathrm{dist}(F, v) = k$.

Claim. For $k \geq 1$, we have $|\mathcal{B}_{C^+}(v)| \leq k$.

Proof. Consider a great-circle $D \neq C$ from $\mathcal{C}$. For a point $x \in C$, we say that (v, x) is D-*separated* if every path from v to x in C^+ intersects D. The set of all D-*separated* points forms a half-circle H_D on C. Let $\mathcal{H}$ be the set of these half-circles, i.e., $\mathcal{H} = \{H_D : D \in \mathcal{C}, D \neq C\}$. See Figure 2.

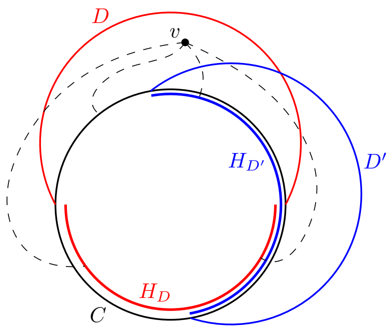

Figure 2 An illustration of the cyclic half-circles $\mathcal{H}$.

We claim that there is a bijection between $\mathcal{B}_{C^+}(v)$ and the $(k-1)$-cliques in $\mathcal{H}$. Indeed, if the intersection of the half-circles of a clique K, viewed as a subset of C, is I_K, then I_K is the interval of C which is reachable from v by crossing the circles corresponding to the half-circles of K. If F is a cell from C^- at distance k from v, then C and a subset of $k-1$ additional circles have to be crossed to reach v from F, i.e., there is a $(k-1)$-clique in $\mathcal{H}$ whose intersection is $F \cap C$. The number of $(k-1)$-cliques in $\mathcal{H}$ is at most k by Lemma 3.2. ◀

Claim. For $k \geq 1$, we have $|\mathcal{F}_k(C^+)| \leq 2e \cdot k(k+1)n$.

Proof. By definition, the set $\mathcal{F}_k(C^+)$ is the set of pairs (F, v) such that $v \in C^+$ is in the $(\leq k)$-zone of C and $F \in \mathcal{B}_{C^+}(v)$. As already noted in Section 1.2, the $(\leq k)$-zone contains at most $4e \cdot (k+1)n$ vertices of $\mathcal{C}$ and at most $2e \cdot (k+1)n$ vertices in C^+. From the above claim we have $|\mathcal{B}_{C^+}(v)| \leq k$, hence we conclude that $|\mathcal{F}_k(C^+)| \leq 2e \cdot k(k+1)n$. ◀

To include the case $k = 0$ we relax the bound to $|\mathcal{F}_k(C^+)| \leq 2e \cdot (k+1)^2 n$. Since C was chosen arbitrarily among all great-circles from $\mathcal{C}$ and C^+ was chosen arbitrarily among the two hemispheres of C, the upper bound from the above claim holds for any induced hemisphere of $\mathcal{C}$. For the union $\mathcal{F}_k$ of the $\mathcal{F}_k(C^+)$ over all the $2n$ choices of the hemisphere C^+, we have

$$|\mathcal{F}_k| \leq \sum_{C^+ \text{ hemisphere}} |\mathcal{F}_k(C^+)| \leq 4e(k+1)^2 n^2.$$

Proof of Theorem 2.1. The k-level with the southpole chosen in cell F consists of the vertices at distance k from F. Thus, the expected complexity of the k-level when choosing F uniformly at random equals $|\mathcal{F}_k|$ divided by the number of cells. Since the number of cells in an arrangement of n great-circles is $2\binom{n}{2} + 2$ and $|\mathcal{F}_k| \leq 4e(k+1)^2 n^2$, we can conclude the statement from

$$\frac{4e \cdot (k+1)^2 \cdot n^2}{2\binom{n}{2} + 2} \leq 4e \cdot (k+1)^2 \cdot \frac{n}{n-1} \leq 4e \cdot (k+2)^2 \cdot \underbrace{\frac{k+1}{k+2} \cdot \frac{n}{n-1}}_{\leq 1}. \qquad \blacktriangleleft$$

4 Discussion

Theorem 2.1 is about arrangements of great-circles. All the elements of the proof, however, carry over to great-pseudocircles whence the result could also be stated for arrangements of great-pseudocircles. Projective arrangements of lines are obtained by antipodal identification from arrangements of great-circles. Hence, if you pick a cell u.a.r. in a projective arrangement of lines (pseudo-lines) the the expected number of vertices at distance k from the cell is as in Theorem 2.1. If the projection Ψ_Π is used to project an arrangements $\mathcal{C}$ of great-pseudocircles to an Euclidean arrangement $\mathcal{L}$ on Π such that the south-poles coincide, then the k-level of $\mathcal{C}$ corresponds to the union of the k- and the $(n-k-2)$-level of $\mathcal{L}$.

With respect to lower bounds we would like to know the answer to:

▶ Question 1. Is there a family of arrangements where the expected size of the middle level is superlinear when the southpole is chosen uniformly at random?

Recursive constructions from [10] and [11] show that the size of the $(n/2 - s)$-level can be in $\Omega(n \log n)$ for any fixed s. Nevertheless computer experiments suggest that if we choose a random southpole for these examples the expected size of the middle level drops to be linear.

Theorem 2.2 deals with the average size of the k-level in arrangements of randomly chosen great-circles. In our model, great-circles are chosen independently and uniformly at random from the sphere. Since point sets, line arrangements, and great-circle arrangements are in strong correspondence the bound from Theorem 2.2 also applies to k-sets in point sets and k-levels of line arrangements from a specific random distribution.

In the context of Erdős–Szekeres-type problems, several articles made use of point sets which are sampled uniformly at random from a convex shape [3, 23, 2, 1]. Also the average size of the convex hull (0-level) is well-studied for sets of points which are sampled uniformly at random from a convex shape K. If K is a disk, the convex hull has expected size $O(n^{1/3})$, and if K is a convex polygon with k sides, the expected size is $O(k \log n)$ [13, 18, 19, 20]. In particular, the expected size of the convex hull is not constant, which is a substantial contrast to our setting. In fact, our setting appears to be closer to the setting of random order types, for which the expected size of the convex hull was recently shown to be $4 + o(1)$

[12]. Hence it would be very interesting to obtain bounds on the average number of k-sets also in this setting. Last but not least, Edelman [7] showed that the expected number of k-sets of an allowable sequence is of order $\Theta(\sqrt{kn})$.

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