Almost-equidistant sets*

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Abstract

For a positive integer d, a set of points in d-dimensional Euclidean space is called *almost-equidistant* if for any three points from the set, some two are at unit distance. Let $f(d)$ denote the largest size of an almost-equidistant set in d-space.

It is known that $f(2) = 7$, $f(3) = 10$, and that the extremal almost-equidistant sets are unique. We have independent, computer-assisted proofs of these statements. It is also known that $f(5) \geq 16$. We further show that $12 \leq f(4) \leq 13$, $f(5) \leq 20$, $18 \leq f(6) \leq 26$, $20 \leq f(7) \leq 34$, and $f(9) \geq f(8) \geq 24$. Up to dimension 7, our work is based on various computer searches, and in dimensions 6 to 9, we have constructions based on the known construction for $d = 5$.

For every dimension $d \geq 3$, we have an example of an almost-equidistant set of $2d + 4$ points in the d-space and we prove the asymptotic upper bound $f(d) \leq O(d^{3/2})$.

1 Introduction and our results

For a positive integer d, we denote the d-dimensional Euclidean space by $\mathbb{R}^d$. A set V of (distinct) points in $\mathbb{R}^d$ is called *almost-equidistant* if among any three of them, some pair is at distance 1. Let $f(d)$ be the maximum size of an almost-equidistant set in $\mathbb{R}^d$. For example, the vertex set of the well-known Moser spindle (Figure 1(a)) is an almost-equidistant set of 7 points in the plane and thus $f(2) \geq 7$.

In this paper we study the growth rate of the function f. We first consider the case when the dimension d is small and give some almost tight estimates on $f(d)$ for $d \leq 9$. Then we turn to higher dimensions and show $2d + 4 \leq f(d) \leq O(d^{3/2})$.

It is trivial that $f(1) = 4$ and that, up to congruence, there is a unique almost-equidistant set on 4 points in $\mathbb{R}$. Bezdek, Naszódi, and Visy [5] showed that an almost-equidistant set in the plane has at most 7 points. Talata (personal communication) showed in 2007 that there is a unique extremal set. We have a simple, computer-assisted proof of this result [3].

* M. Balko has received funding from European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement no. 678765. The authors all acknowledge the support of the ERC Advanced Research Grant no. 267165 (DISCONV). M. Balko and P. Valtr acknowledge support of the grant P202/12/G061 of the Czech Science Foundation (GAČR). M. Scheucher acknowledges support from TBK Automatisierung und Messtechnik GmbH.

34th European Workshop on Computational Geometry, Berlin, Germany, March 21–23, 2018.
This is an extended abstract of a presentation given at EuroCG'18. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

(a) (b)

Figure 1 (a) The Moser spindle. (b) An almost-equidistant set in $\mathbb{R}^3$ on 10 points.

▶ **Theorem 1.1** (Talata, 2007). *The largest number of points in an almost-equidistant set in* $\mathbb{R}^2$ *is* 7*, that is,* $f(2) = 7$*. Moreover, up to congruence, there is only one planar almost-equidistant set with* 7 *points, namely the Moser spindle.*

Figure 1(b) shows an example of an almost-equidistant set of 10 points in $\mathbb{R}^3$. It is made by taking a so-called *biaugmented tetrahedron*, which is a non-convex polytope formed by gluing three unit tetrahedra together at faces, and rotating a copy of it along the axis through the two simple vertices so that two additional unit-distance edges are created. This unit-distance graph is used in a paper of Nechushtan [12] to show that the chromatic number of $\mathbb{R}^3$ is at least 6. Györey [8] showed, by an elaborate case analysis, that this is the unique largest almost-equidistant set in $\mathbb{R}^3$. We have an independent, computer-assisted proof [3].

▶ **Theorem 1.2** (Györey [8]). *The largest number of points in an almost-equidistant set in* $\mathbb{R}^3$ *is* 10*, that is,* $f(3) = 10$*. Moreover, up to congruence, there is only one almost-equidistant set in* $\mathbb{R}^3$ *with* 10 *points.*

In dimension 4, we have only been able to obtain the following bounds.

▶ **Theorem 1.3.** *The largest number of points in an almost-equidistant set in* $\mathbb{R}^4$ *is either* 12 *or* 13*, that is,* $f(4) \in \{12, 13\}$*.*

The lower bound comes from a generalization of the example in Figure 1(b); see also Theorem 1.6. The proofs of the upper bounds in the above theorems are computer assisted. Based on some numerical work to find approximate realisations of graphs, we believe, but cannot prove rigorously, that an almost-equidistant set of 13 points in $\mathbb{R}^4$ does not exist.

▶ **Conjecture 1.4.** *The largest number of points in an almost-equidistant set in* $\mathbb{R}^4$ *is* 12*.*

In dimension 5, Larman and Rogers [11] showed that $f(5) \geq 16$ by a construction based on the so-called Clebsch graph. In dimensions 6 to 9, we use their construction to obtain lower bounds that are stronger than the lower bound $2d + 4$ stated below in Theorem 1.6. We again complement this with some computer-assisted upper bounds.

▶ **Theorem 1.5.** *The largest number of points in an almost-equidistant set in* $\mathbb{R}^5$, $\mathbb{R}^6$, $\mathbb{R}^7$, $\mathbb{R}^8$ *and* $\mathbb{R}^9$ *satisfy the following:* $16 \leq f(5) \leq 20$, $18 \leq f(6) \leq 26$, $20 \leq f(7) \leq 34$, $24 \leq f(8) \leq 41$, *and* $24 \leq f(9) \leq 49$*.*

The *unit-distance graph* of an almost-equidistant point set P in $\mathbb{R}^d$ is obtained from P by letting P be its vertex set and by placing an edge between pairs of points at unit distance.

For every $d \in \mathbb{N}$, a unit-distance graph in $\mathbb{R}^d$ does not contain K_{d+2} (see Corollary 2.2) and the complement of the unit-distance graph of an almost-equidistant set is triangle-free.

Dimension d	1	2	3	4	5	6	7	8	9	$d \geq 9$
Lower bounds on $f(d)$	4	7	10	12	16	18	20	24	24	$2d+4$
Upper bounds on $f(d)$	4	7	10	13	20	26	34	41	49	$4(d^{3/2}+\sqrt{d})$

Table 1 Lower and upper bounds on the largest size of an almost-equidistant set in $\mathbb{R}^d$.

Thus we have $f(d) \leq R(d+2,3)-1$, where $R(d+2,3)$ is the *Ramsey number* of K_{d+2} and K_3, that is, the smallest positive integer N such that for every graph G on N vertices there is a copy of K_{d+2} in G or a copy of K_3 in the complement of G.

Ajtai, Komlós, and Szemerédi [1] showed $R(d+2,3) \leq O(d^2/\log d)$ and this bound is known to be tight [9]. We thus have an upper bound $f(d) \leq O(d^2/\log d)$, which, as we show below, is not tight. For small values of d where the Ramsey number $R(d+2,3)$ is known or has a reasonable upper bound, we obtain an upper bound for $f(d)$. In particular, we get $f(5) \leq 22$, $f(6) \leq 27$, $f(7) \leq 35$, $f(8) \leq 41$, and $f(9) \leq 49$ [16]. For $d \in \{5,6,7\}$, we slightly improve these estimates to the bounds from Theorem 1.5 using our computer-assisted approach [3].

We now turn to higher dimensions. The obvious generalization of the Moser spindle gives an example of an almost-equidistant set of $2d+3$ points in $\mathbb{R}^d$. The next theorem improves this by 1. It is a generalization of the almost-equidistant set on 10 points in $\mathbb{R}^3$ from Figure 1(b).

▶ **Theorem 1.6.** *For every $d \geq 3$, there is an almost-equidistant set in $\mathbb{R}^d$ with $2d+4$ points.*

Rosenfeld [17] showed that an almost-equidistant set on a sphere in $\mathbb{R}^d$ of radius $1/\sqrt{2}$ has size at most $2d$, which is best possible. Rosenfeld's proof, which uses linear algebra, was adapted by Bezdek and Langi [4] to spheres of other radii. They showed that an almost-equidistant set on a sphere in $\mathbb{R}^d$ of radius $\leq 1/\sqrt{2}$ has at most $2d+2$ elements, which is attained by the union of two d-simplices inscribed in the same sphere.

Pudlák [15] and Deaett [6] gave simpler proofs of Rosenfeld's result. Our final result is an asymptotic upper bound for the size of an almost-equidistant set, based on Deaett's proof [6].

▶ **Theorem 1.7.** *An almost-equidistant set of points in $\mathbb{R}^d$ has cardinality $O(d^{3/2})$.*

We note that Polyanskii [13] recently found an upper bound of $O(d^{13/9})$ for the size of an almost-equidistant set in $\mathbb{R}^d$ and Kupavskii, Mustafa, and Swanepoel [10] and Polyanskii [14] improved this to $O(d^{4/3})$. Both papers use ideas from our proof of Theorem 1.7.

In this paper, we use $\|v\|$ to denote the Euclidean norm of a vector v from $\mathbb{R}^d$. For a subset S of $\mathbb{R}^d$, we use $\mathrm{span}(S)$ to denote the linear hull of S.

In the rest of the paper we sketch the proof of Theorem 1.7. The proofs of the remaining statements, as well as some auxiliary claims, can be found in the full version of this paper [3]. The full version also contains a computer program that enumerates all graphs that are unit-distance graphs of almost-equidistant sets up to a certain size and dimension. The source code of our programs and the files are available on a separate website [18].

2 Proof of Theorem 1.7

In this section, we sketch the proof of Theorem 1.7 by showing the upper bound $f(d) \leq O(d^{3/2})$. As a first step towards this proof, we state the following lemma that characterizes sets of points lying at the unit distance from vertices of a regular simplex with unit-length edges. For the statement of the lemma, we recall that a sphere of dimension d is a surface of a $(d+1)$-dimensional ball.

▶ **Lemma 2.1.** *For $d, k \in \mathbb{N}$, let C be a set of k points in $\mathbb{R}^d$ such that the distance between any two of them is 1. Let $c := \frac{1}{k}\sum_{p\in C} p$ be the centroid of C and let $A := \operatorname{span}(C - c)$. Then the set of points equidistant from all points of C is the affine space $c + A^{\perp}$ orthogonal to A and passing through c. Furthermore, the intersection of all unit spheres centred at the points in C is the $(d-k)$-dimensional sphere of radius $\sqrt{(k+1)/(2k)}$ centred at c and contained in $c + A^{\perp}$.*

▶ **Corollary 2.2.** *For $d \in \mathbb{N}$, every subset of $\mathbb{R}^d$ contains at most $d+1$ points that are pairwise at unit distance.*

The following lemma is a well-known result that bounds the rank of a square matrix from below in terms of the entries of the matrix [2, 6, 15].

▶ **Lemma 2.3.** *Let $A = [a_{i,j}]$ be a non-zero symmetric $m \times m$ matrix with real entries. Then*

$$\operatorname{rank} A \geq \Big(\sum_{i=1}^{m} a_{i,i}\Big)^2 \Big/ \sum_{i=1}^{m}\sum_{j=1}^{m} a_{i,j}^2.$$

The last lemma before the proof of Theorem 1.7 can be proved by a calculation, using its assumption that the vectors v_i have pairwise inner products ε, so they differ from an orthogonal set by some skewing.

▶ **Lemma 2.4.** *For $n, t \in \mathbb{N}$ with $t \leq n$, let $w_1, \dots, w_t$ be unit vectors in $\mathbb{R}^n$ such that $\langle w_i, w_j\rangle = \varepsilon$ for all i, j with $1 \leq i < j \leq t$, where $\varepsilon \in [0, 1)$. Then the set $\{w_1, \dots, w_t\}$ can be extended to $\{w_1, \dots, w_n\}$ such that $\langle w_i, w_j\rangle = \varepsilon$ for all i, j with $1 \leq i < j \leq n$, and such that for some orthonormal basis $e_1, \dots, e_n$ we have $w_i = \frac{e_i + \lambda e}{\|e_i + \lambda e\|}$ $(i = 1, \dots, n)$, where*

$$\lambda := \frac{-1 + \sqrt{1 + \varepsilon n/(1-\varepsilon)}}{n} \quad \textit{and} \quad e := \sum_{j=1}^{n} e_j = \frac{1}{\sqrt{1 + (n-1)\varepsilon}} \sum_{j=1}^{n} w_j.$$

Moreover, $\|e_i + \lambda e\|^2 = (1-\varepsilon)^{-1}$ for each $i \in \{1, \dots, n\}$ and for every $x \in \mathbb{R}^n$ we have

$$\sum_{j=1}^{n} (\langle x, w_j\rangle - \varepsilon)^2 = (1-\varepsilon)(\|x\|^2 - \varepsilon) + \varepsilon\left(\langle x, e\rangle - \sqrt{1 + (n-1)\varepsilon}\right)^2.$$

We are now ready to prove Theorem 1.7. For $d \geq 2$, let $V \subset \mathbb{R}^d$ be an almost-equidistant set. Let $G = (V, E)$ be the unit-distance graph of V and let $k := \lfloor 2\sqrt{d}\rfloor$. Note that $2 \leq k \leq d$.

Let $S \subseteq V$ be a set of k points such that the distance between any two of them is 1. If such a set does not exist, then, since the complement of G does not contain a triangle, we have $|V| < R(k, 3)$, where $R(k, 3)$ is the Ramsey number of K_k and K_3. Using the bound $R(k, 3) \leq \binom{k+3-2}{3-1}$ obtained by Erdős and Szekeres [7], we derive $|V| < \binom{2\sqrt{d}+1}{2} = 2d + \sqrt{d}$. Thus we assume in the rest of the proof that S exists.

Let B be the set of common neighbours of S, that is, $B := \{x \in V \mid \|x - s\| = 1 \ \forall s \in S\}$. Since V is almost-equidistant, the set of non-neighbours of any vertex of G is a clique and so it has size at most $d + 1$ by Corollary 2.2. Every vertex from $V \setminus B$ is a non-neighbour of some vertex from S and thus it follows that $|V \setminus B| \leq k(d + 1)$.

We now estimate the size of B. By Lemma 2.1 applied to S, the set B lies on a sphere of radius $\sqrt{(k+1)/2k}$ in an affine subspace of dimension $d - k + 1$. We may take the centre of this sphere as the origin, and rescale by $\sqrt{2k/(k+1)}$ to obtain a set B' of m unit vectors $v_1, \dots, v_m \in \mathbb{R}^{d-k+1}$ where $m := |B|$. For any three of the vectors from B', the distance between some two of them is $\sqrt{2k/(k+1)}$. For two such vectors v_i and v_j with $\|v_i - v_j\|^2 = 2k/(k+1)$, the facts $\|v_i - v_j\|^2 = \|v_i\|^2 + \|v_j\|^2 - 2\langle v_i, v_j\rangle$ and $\|v_i\|^2 = \|v_j\|^2 = 1$ imply $\langle v_i, v_j\rangle = \varepsilon$, where $\varepsilon := 1/(k+1)$. Note that the opposite implication also holds. That is, if $\langle v_i, v_j\rangle = \varepsilon$, then v_i and v_j are at distance $\sqrt{2k/(k+1)}$.

Let $A = [a_{i,j}]$ be the $m \times m$ matrix defined by $a_{i,j} := \langle v_i, v_j\rangle - \varepsilon$. Clearly, A is a symmetric matrix with real entries. If $m \geq d-k+2$, then A is also non-zero, as G contains no K_{d+2} and every vertex from B is adjacent to every vertex from S in G. We recall that $\operatorname{rank} XY \leq \min\{\operatorname{rank} X, \operatorname{rank} Y\}$ and $\operatorname{rank}(X+Y) \leq \operatorname{rank} X + \operatorname{rank} Y$ for two matrices X and Y. Since $B' = \{v_1, \ldots, v_m\} \subset \mathbb{R}^{d-k+1}$ and

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}^\top \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} - \varepsilon J,$$

where J is the $m \times m$ matrix with each entry equal to 1, we have

$$\operatorname{rank} A \leq d-k+2. \tag{1}$$

By Lemma 2.3,

$$\operatorname{rank} A \geq \frac{\left(\sum\limits_{i=1}^{m} a_{i,i}\right)^2}{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{m} a_{i,j}^2} = \frac{m^2(1-\varepsilon)^2}{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{m}(\langle v_i, v_j\rangle - \varepsilon)^2}. \tag{2}$$

For $i \in \{1, \ldots, m\}$, let N_i be the set of vectors from B' that are at distance $\sqrt{2k/(k+1)}$ from v_i. That is, $N_i := \{v_j \in B' \mid \langle v_i, v_j\rangle = \varepsilon\}$. Then for each fixed v_i we have

$$\sum_{j=1}^{m}(\langle v_i, v_j\rangle - \varepsilon)^2 = (1-\varepsilon)^2 + \sum_{v_j \in N_i} 0 + \sum_{v_j \in B' \setminus (N_i \cup \{v_i\})} (\langle v_i, v_j\rangle - \varepsilon)^2. \tag{3}$$

Note that the vectors from $B' \setminus (N_i \cup \{v_i\})$ have pairwise inner products ε, as neither of them is at distance $\sqrt{2k/(k+1)}$ from v_i, and thus $|B' \setminus (N_i \cup \{v_i\})| \leq d-k+2$. In fact, we even have $|B' \setminus (N_i \cup \{v_i\})| \leq d-k+1$, since B' contains only unit vectors and any subset of $d-k+2$ points from B' with pairwise distances $\sqrt{2k/(k+1)}$ would form the vertex set of a regular $(d-k+1)$-simplex with edge lengths $\sqrt{2k/(k+1)}$ centred at the origin. However, then the distance from the centroid of such a simplex to its vertices would be equal to $\sqrt{k(d-k+1)/((k+1)(d-k+2))} \neq 1$, which is impossible.

Thus setting $n := d-k+1$ and $t := |B' \setminus (N_i \cup \{v_i\})|$, we have $t \leq n$. Applying Lemma 2.4 to the t vectors from $B' \setminus (N_i \cup \{v_i\}) \subseteq \mathbb{R}^n$ with $\varepsilon = (k+1)^{-1}$ and $x = v_i$, we see that the last sum in (3) is at most

$$(1-\varepsilon)^2 + \varepsilon\left(\langle v_i, e\rangle - \sqrt{1+(d-k)\varepsilon}\right)^2,$$

where $e = \sum_{j=1}^{d-k+1} e_j$ for some orthonormal basis $e_1, \ldots, e_{d-k+1}$ of $\mathbb{R}^{d-k+1}$.

By the Cauchy–Schwarz inequality,

$$\begin{aligned}\left(\langle v_i, e\rangle - \sqrt{1+(d-k)\varepsilon}\right)^2 &< \left(\sqrt{d-k+1} + \sqrt{1+(d-k)\varepsilon}\right)^2 \\ &= d-k+1+2\sqrt{d-k+1}\sqrt{1+(d-k)\varepsilon}+1+(d-k)\varepsilon < 4(d-k+1).\end{aligned}$$

Recall that $k \geq 2$. Using $\varepsilon = (k+1)^{-1}$, we obtain

$$\sum_{j=1}^{m}\big(\langle v_i, v_j\rangle - \varepsilon\big)^2 < 2(1-\varepsilon)^2 + 4\varepsilon(d-k+1) = 4\varepsilon d + 2(1+\varepsilon)^2 - 4 < 4\varepsilon d.$$

If we substitute this upper bound back into (2), then with (1) we obtain that $d-k+2 > m^2(1-\varepsilon)^2/(4m\varepsilon d)$ and thus $m < (4\varepsilon d)(d-k+2)/(1-\varepsilon)^2$. Using the choice $k = \lfloor 2\sqrt{d} \rfloor$ and the expression $\varepsilon = (k+1)^{-1}$, we obtain $(d-k+2)/(1-\varepsilon)^2 < d$, if $d \geq 8$, and thus $m < 4d^2/(k+1)$. Altogether, we have $m \leq \max\{d-k+1, 4d^2/(k+1)\} = 4d^2/(k+1)$. It follows that $|V| \leq k(d+2)+4d^2/(k+1)$. Again, using the choice $k = \lfloor 2\sqrt{d} \rfloor \in (2\sqrt{d}-1, 2\sqrt{d}]$, we conclude that $|V| < 2\sqrt{d}(d+2) + 4d^2/(2\sqrt{d}) = 4d^{3/2} + 4\sqrt{d}$. This finishes the proof of Theorem 1.7.

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