Topological Drawings meet Classical Theorems from Convex Geometry*

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Abstract

In this article, we discuss classical theorems from Convex Geometry in the context of simple topological drawings of the complete graph K_n. In a simple topological drawing, any two edges share at most one point: either a common vertex or a point where they cross.

We present generalizations of Kirchberger's Theorem and the Erdős–Szekeres Theorem, a family of simple topological drawings with arbitrarily large Helly number, a new proof of the generalized Carathéodory's Theorem, and discuss further classical theorems from Convex Geometry in the context of simple topological drawings.

1 Introduction

A point set in the plane (in general position) induces a straight-line drawing of the complete graph K_n. In this article we investigate topological drawings of K_n and use the triangles of such drawings for convexity related studies. In a *topological drawing* D of K_n, vertices are mapped to points in the plane and edges are mapped to simple curves connecting the corresponding end-points such that every pair of edges has at most one common point, which is either a common endpoint or a crossing; see Figure 1. Note that several edges may cross in a single point. A topological drawing is called *straight-line* if all edges are drawn as line segments, and *pseudolinear* if all arcs of the drawing can be extended to bi-infinite curves such that any two of these curves cross at most once (the family of curves is an arrangement of pseudolines).

Figure 1 Forbidden patterns in topological drawings: self-crossings, double-crossings, touchings, and crossings of adjacent edges.

1.1 Our Results

In Section 2, we present a new combinatorial generalization of topological drawings – which we name *generalized signotopes*. They allow us to prove a generalization of Kirchberger's

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Theorem in Section 3. We present a family of topological drawings with arbitrarily large Helly number in Section 4, and, in Section 5, we discuss generalizations of Carathéodory's Theorem, Erdős–Szekeres Theorem, and colorful variants of the classical theorems.

In the full version, we also describe the SAT model, which we used to test hypotheses about simple topological drawings, and give a more detailed analysis of the named theorems in terms of the *convexity* hierarchy introduced by Arroyo, McQuillan, Richter, and Salazar [1]. In particular, some of the theorems turn out to generalize to pseudolinear drawings but not to *pseudocircular* drawings, i.e., topological drawings where all arcs can be simultaneously extended to pseudocircles such that any two do not touch and cross at most twice.

2 Preliminaries

Given a topological drawing D of K_n, we call the induced subdrawing of three vertices a *triangle*. Note that the edges of a triangle in a topological drawing do not cross. The removal of a triangle separates the plane into two connected components. A point p is in the *interior* of a triangle or more generally of a topological drawing D if p is in a bounded connected component of $\mathbb{R}^2 - D$.

In a topological drawing, we assign an *orientation* $\chi(abc) \in \{+, -\}$ to each ordered triple abc of vertices. The sign $\chi(abc)$ indicates whether we go counterclockwise or clockwise around the triangle when visiting the vertices a, b, c in this order.

In a straight-line drawing of K_n the underlying point set $S = \{s_1, \ldots, s_n\}$ is in general position (no three points lie on a line). If the points are sorted from left to right, then for every 4-tuple s_i, s_j, s_k, s_l with $i < j < k < l$ the sequence $\chi(ijk), \chi(ijl), \chi(ikl), \chi(jkl)$ (index-triples in lexicographic order) is *monotone*, i.e., there is at most one sign-change. A *signotope* is a mapping $\chi : \binom{[n]}{3} \to \{+, -\}$ with the above monotonicity property. Signotopes are in bijection with Euclidean pseudoline arrangements and can be used to characterize pseudolinear drawings [7, 3].

Let us now consider topological drawings of the complete graph. There are two types of drawings of K_4 on the sphere: type I has a crossing and type II has no crossing. Type I can be drawn in two different ways in the plane: in type I_a the crossing is only incident to bounded faces and in type I_b the crossing lies on the outer face; see Figure 2.

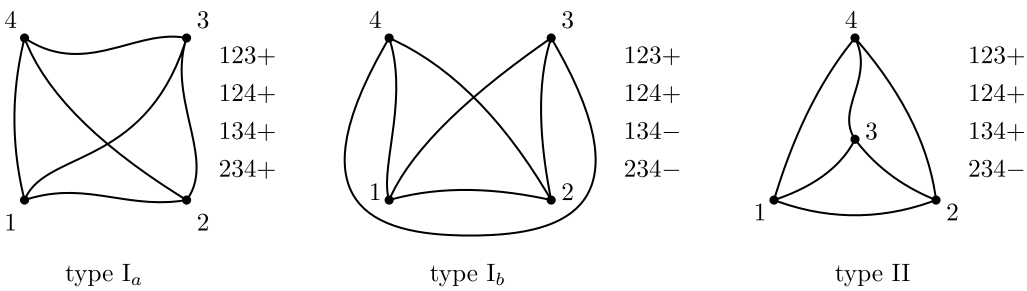

Figure 2 The three types of topological drawings of K_4 in the plane.

A drawing of K_4 with vertices a, b, c, d can be characterized in terms of the sequence of orientations $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$. The drawing is

- of type I_a or type I_b iff the sequence is $++++$, $++--$, $+--+$, $-++-$, $--++$, or $----$; and
- of type II iff the number of $+$'s (and $-$'s respectively) in the sequence is odd.

Therefore there are at most two sign-changes in the sequence and, moreover, any such sequence is in fact induced by a topological drawing of K_4. Allowing up to two sign-changes is equivalent to forbidding the two patterns $+-+-$ and $-+-+$.

If a mapping $\chi : [n]_3 \to \{+,-\}$ is *alternating*, i.e., $\chi(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}) = \text{sgn}(\sigma) \cdot \chi(i_1, i_2, i_3)$, and χ avoids the two patterns on sorted indices, i.e., $\chi(ijk), \chi(ijl), \chi(ikl), \chi(jkl)$ has at most two sign-changes for $i < j < k < l$, then it avoids the two patterns on $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$ for any pairwise different $a, b, c, d \in [n]$. We refer to this as the *symmetry property* of the forbidden patterns.

The symmetry property allows us to define *generalized signotopes* as alternating mappings $\chi : [n]_3 \to \{+,-\}$ with at most two sign-changes on $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$ for any pairwise different $a, b, c, d \in [n]$. We remark that generalized signotopes can be seen as a proper generalization of topological drawings of K_n – details are deferred to the full version.

3 Kirchberger's Theorem

Two point sets $A, B \subseteq \mathbb{R}^d$ are called *separable* if there exists a hyperplane H separating them. It is well-known that if two sets A, B are separable they can also be separated by a hyperplane H containing one point of A and B. *Kirchberger's Theorem* (see [12] or [5]) asserts that two finite point sets $A, B \subseteq \mathbb{R}^d$ are separable if and only if for every $C \subseteq A \cup B$ with $|C| = d + 2$, $C \cap A$ and $C \cap B$ are separable.

We prove a generalization of the 2-dimensional version of Kirchberger's Theorem in the setting of generalized signotopes, where two sets $A, B \subseteq [n]$ are *separable* if there exist $i, j \in A \cup B$ such that $\chi(i, j, x) = +$ for all $x \in A \setminus \{i, j\}$ and $\chi(i, j, x) = -$ for all $x \in B \setminus \{i, j\}$. In this case we say that ij *separates* A from B and write $\chi(i, j, A) = +$ and $\chi(i, j, B) = -$. Moreover, if we can find $i \in A$ and $j \in B$, we say that A and B are *strongly separable*.

▶ **Theorem 3.1** (Kirchberger's Theorem for Generalized Signotopes)**.** *Let $\chi : [n]_3 \to \{+,-\}$ be a generalized signotope, and $A, B \subseteq [n]$. If for every $C \subseteq A \cup B$ with $|C| = 4$, the sets $A \cap C$ and $B \cap C$ are (strongly) separable, then A and B are (strongly) separable.*

Proof. First note that it is sufficient to prove the theorem for strongly separable since all 4-tuples which are weakly separable are also strongly separable. More details will be given in the full version. Hence in the following we always assume that the 4-tuples are strongly separable.

To prove the claim we consider a counterexample (χ, A, B) minimizing the size of the smaller of the two sets. By symmetry we may assume $|A| \leq |B|$. First we consider the cases $|A| = 1, 2, 3$ individually and then the case $|A| \geq 4$.

Let $A = \{a\}$, we need to find $b \in B$ such that $\chi(a, b, B) = -$. Let B' be a maximal subset of B such that B' can be separated from $\{a\}$, and let $b \in B$ be such that $\chi(a, b, B') = -$. Suppose that $B' \neq B$, then there is a $b^* \in B \setminus B'$ with

$$\chi(a, b, b^*) = +. \tag{1}$$

By maximality of B', the sets $\{a\}$ and $B' \cup \{b^*\}$ cannot be separated. Hence, we have

$$\chi(a, b^*, b') = + \tag{2}$$

for some $b' \in B'$. Since χ is alternating (1) and (2) together imply $b' \neq b$. Since $b' \in B'$ we have $\chi(a, b, b') = -$. From this together with (1) and (2) it follows that the four element set $\{a, b, b', b^*\}$ has no separator. This is a contradiction, whence $B' = B$.

As a consequence we obtain:

- Every one-element set $\{a\}$ with $a \in A$ can be separated from B. Since χ is alternating there can be at most one $b(a) \in B$ such that $\chi(a, b(a), B) = -$.

Now we look at the case where $A = \{a_1, a_2\}$. Let $b_i = b(a_i)$, if $b_1 = b_2$, or $\chi(a_1, b_1, a_2) = +$, or $\chi(a_2, b_2, a_1) = +$ we have a separator for A and B. So assume that $b_1 \neq b_2$, and $\chi(a_1, b_1, a_2) = -$, and $\chi(a_2, b_2, a_1) = -$. Since χ is alternating we also know that $\chi(a_1, b_2, b_1) = +$ and $\chi(a_2, b_1, b_2) = +$. Together these four signs show that $\{a_1, b_1, a_2, b_2\}$ is not separable, a contradiction.

The case $|A| = 3$ works similarly but is more technical. A proof of this case, will be given in the full version.

Now we consider the remaining case where (χ, A, B) is a minimal counterexample with $4 \leq |A| \leq |B|$.

Let $a^* \in A$. By minimality of (χ, A, B), $A \backslash \{a^*\} = A'$ is separable from B. Let $a \in A'$ and $b \in B$ such that $\chi(a, b, A') = +$ and $\chi(a, b, B) = -$. Hence it is

$$\chi(a, b, a^*) = -. \tag{3}$$

Let $b^* = b(a^*)$, i.e., $\chi(a^*, b^*, B) = -$. There is some $a' \in A'$ such that

$$\chi(a^*, b^*, a') = -. \tag{4}$$

If $a' = a$, then $b \neq b^*$ because of (3) and (4). From $\chi(a, b, B) = -$, $\chi(a^*, b^*, B) = -$, (3), and (4) it follows that the 4-element set $\{a, b, a^*, b^*\}$ has no separation. The contradiction shows $a' \neq a$.

Let $b' = b(a')$. If $b = b'$, then $a' \in A'$ implies $\chi(a, b, a') = +$ which yields $\chi(a', b', a) = -$. If $b \neq b'$ we look at the four elements a, b, a', b', and have:

$$\chi(a', b', a) =? \ , \quad \chi(a', b', b) = - \ , \quad \chi(a', a, b) = + \ , \quad \chi(b', a, b) = -$$

To avoid the forbidden pattern for $a'b'ab$ we must have $\chi(a', b', a) = -$.

Hence, regardless whether $b = b'$ or $b \neq b'$ we have

$$\chi(a', b', a) = - \ . \tag{5}$$

Since $|A| \geq 4$, we know by the minimality of (χ, A, B) that the set $\{a, b, a', b', a^*, b^*\}$, which has 3 elements of A and at least 4 elements in total, is separable. It follows from $\chi(a, b, B) = \chi(a', b', B) = \chi(a^*, b^*, B) = +$ that the only possible separators are ab, $a'b'$, and a^*b^*. They, however, do not separate because of (3), (5), and (4) respectively. This is impossible, hence, there is no counterexample. ◀

4 Helly's Theorem

Helly's Theorem asserts that the intersection of n convex sets $S_1, \ldots, S_n$ in $\mathbb{R}^d$ is non-empty if the intersection of every $d+1$ of these sets in non-empty. In other words, the *Helly number* of a family of n convex sets in $\mathbb{R}^d$ is at most $d+1$. The result of Goodman and Pollack [9] (see also [2]) shows that Helly's Theorem holds for pseudoconfigurations of points in two dimensions, and thus for pseudolinear drawings.

In the more general setting of topological drawings, we investigate the intersection properties of triangles. We show that Helly's Theorem does not generalize to topological drawings by constructing topological drawings with arbitrarily large Helly number. Note that the following theorem does not contradict the topological Helly Theorem [10] (cf. [8]) because the intersections of two triangles are not connected.

▶ **Theorem 4.1.** *For every odd integer n, there exists a topological drawing of K_{3n} with Helly number at least n, i.e., there are n triangles such that any $n-1$ have a common interior, but not all n have a common interior.*

Sketch of the proof. The basic idea of the construction is to draw n triangles on the cylinder with the property that any $n-1$ triangles have a common interior, while there is no common intersection of all n triangles; the left-hand side of Figure 3 gives an illustration. Such a cylindrical drawing can clearly be drawn in the plane as depicted on the right-hand side of Figure 3. The technical part, however, is to complete such a drawing of n triangles to a topological drawing of the complete graph K_{3n}. Technical details are deferred to the full version.

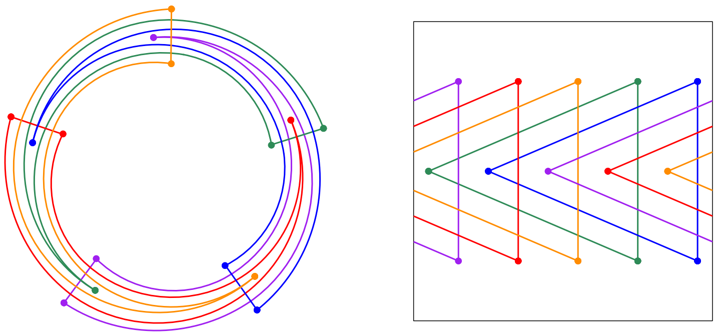

Figure 3 An illustration of a topological drawing of K_{3n} with Helly number n.

For the construction, we identify the cylinder surface with the real plane $\mathbb{R}^2$. As illustrated in Figure 4, we place the vertices of K_{3n} on 3 different layers.

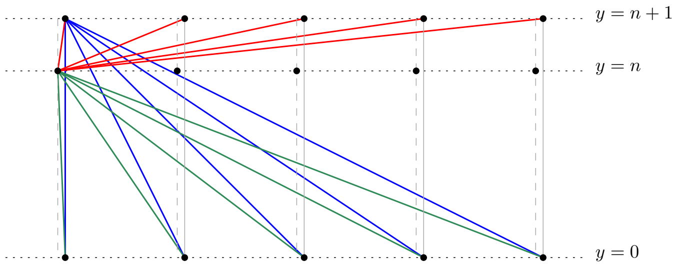

Figure 4 Placement of the vertices and drawing edges between different layers.

The edges between different layers are drawn as straight-line segments (see Figure 4). Edges between two vertices of the same layer are drawn as circular arcs above the top layer, below the bottom layer and in a sufficiently small range above the middle layer (see Figure 5). The obtained drawing is topological (details will be given in the full version), which concludes the proof. ◀

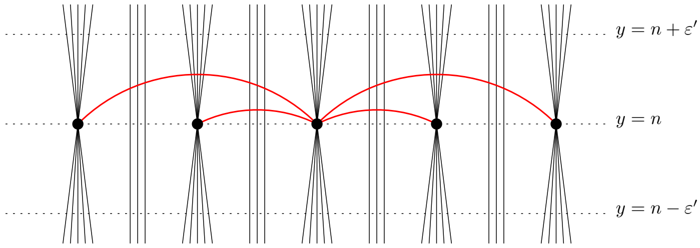

Figure 5 Edges between middle-layer vertices are drawn as very flat circular arcs.

5 Further Results

Carathéodory's Theorem asserts that, if a point x lies in the convex hull of a point set P in $\mathbb{R}^d$, then x lies in the convex hull of at most $d+1$ points of P. A more general version of Carathéodory's Theorem in the plane is by Balko, Fulek, and Kynčl, who provided a generalization to topological drawings [3, Lemma 4.7]. In the full version, we present a new proof for their theorem.

▶ **Theorem 5.1** (Carathéodory for Topological Drawings [3])**.** *Let D be a topological drawing of K_n and let $x \in \mathbb{R}^2$ be a point in the interior of D. Then there is a triangle in D which contains x in its interior.*

In the full version, we also study colorful variants of the theorems. For example it turned out that Barany's Colorful Carathéodory Theorem [4] holds up to pseudolinear drawings [11] but not for pseudocircular drawings (see Figure 6).

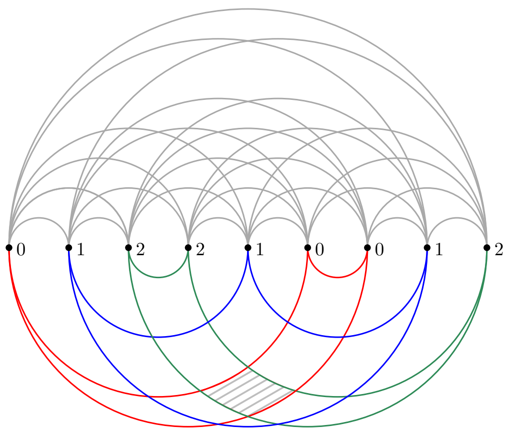

Figure 6 A circular drawing of K_9 violating the condition of Colorful Carathéodory Theorem.

The classical *Erdős–Szekeres Theorem* [6] asserts that every straight-line drawing of K_n contains a crossing-maximal subdrawing of size $k = \Omega(\log n)$, that is, a subdrawing of K_k with $\binom{k}{4}$ crossings. Pach, Solymosi, and Tóth [13] generalized this result by showing that every topological drawing of K_n has a crossing-maximal subdrawing of size $k = \Omega(\log^{1/8} n)$. A simple Ramsey–type argument shows that a variant of the Erdős–Szekeres Theorem even applies to generalized signotopes.

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