

Topological Drawings meet Classical Theorems from Convex Geometry*

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Abstract

In this article we discuss classical theorems from Convex Geometry in the context of topological drawings. In a simple topological drawing of the complete graph K_n , any two edges share at most one point: either a common vertex or a point where they cross. Triangles of simple topological drawings can be viewed as convex sets, this gives a link to convex geometry.

We present a generalization of Kirchner's Theorem, a family of simple topological drawings with arbitrarily large Helly number, and a new proof of a topological generalization of Carathéodory's Theorem in the plane. We also discuss further classical theorems from Convex Geometry in the context of simple topological drawings.

We introduce “generalized signotopes” as a generalization of topological drawings. As indicated by the name they are a generalization of signotopes, a structure studied in the context of encodings for arrangements of pseudolines.

1 Introduction

A point set in the plane (in general position) induces a straight-line drawing of the complete graph K_n . In this article we investigate topological drawings of K_n and use the triangles of such drawings to generalize and study classical problems from the convex geometry of point sets. In a *topological drawing* D of K_n ,

- ▶ vertices are mapped to distinct points in the plane,
- ▶ edges are mapped to simple curves connecting the two corresponding vertices and containing no other vertices, and

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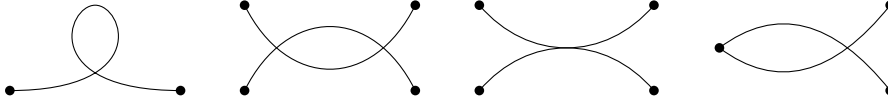


Figure 1: Forbidden patterns in topological drawings: self-crossings, double-crossings, touchings, and crossings of adjacent edges.

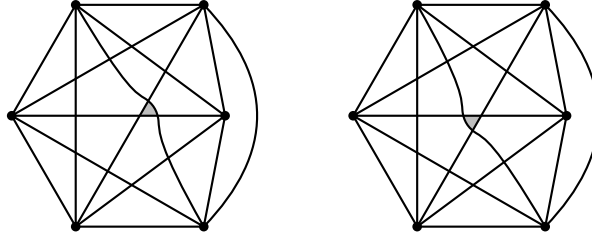


Figure 2: Two weakly isomorphic drawings of K_6 , which can be transformed into each other by a triangle-flip.

- every pair of edges has at most one common point, which is either a common vertex or a crossing (but not a touching).

Figure 1 illustrates the forbidden patterns for topological drawings. Moreover, we assume throughout the article that no three or more edges cross in a single point. Topological drawings are also known as “good drawings” or “simple drawings”.

In this article, we discuss classical theorems such as Kirchberger’s, Helly’s, and Carathéodory’s Theorem in terms of the *convexity* hierarchy of topological drawings introduced by Arroyo, McQuillan, Richter, and Salazar [AMRS17a], which we introduce in Section 2. In this section, we also introduce *generalized signotopes*, a combinatorial generalization of topological drawings. Our proof of a generalization of Kirchberger’s Theorem in Section 3 makes use of this structure. Section 4 deals with a generalization of Carathéodory’s Theorem. In Section 5, we present a family of topological drawings with arbitrarily large Helly number. We conclude this article with Section 6, where we discuss some open problems.

2 Preliminaries

Let D be a topological drawing and v a vertex of D . The cyclic order π_v of incident edges around v is called the *rotation* of v in D . The collection of rotations of all vertices is called the *rotation system* of D . Two topological drawings are *weakly isomorphic* if there is an isomorphism of the underlying abstract graphs which preserves the rotation system or reverses all rotations.

A triangular cell, which has no vertex on its boundary, is bounded by three edges. By moving one of these edges across the intersection of the two other edges, one obtains a weakly isomorphic drawing; see Figure 2. This operation is called *triangle-flip*. Gioan [Gio05], see also Arroyo et al. [AMRS17b], showed that two weakly isomorphic drawings of the complete graph can be transformed into each other with a sequence of triangle-flips and reflections of the drawing.

Besides weak isomorphism, there is also the notion of strong isomorphism: two topological drawings are called *strongly isomorphic* if they induce homeomorphic cell decompositions of the sphere. Every two strongly isomorphic drawings are also weakly isomorphic.

Convexity Hierarchy. Given a topological drawing D , we call the induced subdrawing of three vertices a *triangle*. Note that the edges of a triangle in a topological drawing do not cross. The removal of a triangle separates the plane into two connected components – a bounded component and an unbounded component. We call the closure of these connected components *sides*. A side of a triangle is *convex* if every edge which has its two end-vertices in the side is completely drawn in the side. We are now ready to introduce the “convexity-hierarchy” of Arroyo et al. [AMRS17a]). For $k = 1, \dots, 5$, if \mathcal{D}_k denotes the class of drawings with property (k) from the list below, then $\mathcal{D}_{k+1} \subsetneq \mathcal{D}_k$.

- (1) topological drawings;
- (2) *convex* drawings: each triangle has a convex side;
- (3) *hereditary-convex* drawings: if a triangle \triangle_1 is fully contained in the convex side of another triangle \triangle_2 , then also its convex side is;
- (4) *face-convex* drawings: there is a special face f_∞ such that, for every triangle, the side not containing f_∞ is convex;
- (5) *pseudolinear* drawings: all edges of the drawing can be extended to bi-infinite curves – called *pseudolines* – such that any two cross at most once¹;
- (6) straight-line drawings: all edges are drawn as straight-lines connecting their endpoints.

Arroyo et al. [AMRS18] showed that the face-convex drawings where the special face f_∞ is drawn as the unbounded outer face are precisely the pseudolinear drawings (see also [ABR20] and [AHP⁺15]).

Pseudolinear drawings are generalized by pseudocircular drawings. A drawing is called *pseudocircular* if the edges can be extended to pseudocircles (simple closed curves) such that any pair of non-disjoint pseudocircles has exactly two crossings. Since stereographic projections preserve (pseudo)circles, pseudocircularity is a property of drawings on the sphere.

Pseudocircular drawings were studied in a recent article by Arroyo, Richter, and Sunohara [ARS20]. They provided an example of a topological drawing which is not pseudocircular. Moreover, they proved that hereditary-convex drawings are precisely the *pseudospherical* drawings, i.e., pseudocircular drawings with the additional two properties that

- ▶ every pair of pseudocircles intersects, and
- ▶ for any two edges $e \neq f$ the pseudocircle γ_e has at most one crossing with f .

The relation between convex drawings and pseudocircular drawings remains open.

Convexity, hereditary-convexity, and face-convexity are properties of the weak isomorphism classes. To see this note that the existence of a convex side is not affected by changing the outer face or transferring the drawing to the sphere, moreover, convex sides are not affected by triangle-flips. Hence, these properties only depend on the rotation system of the drawing. For pseudolinear and straight-line drawings, however, the choice of the outer face plays an essential role.

¹Arrangements of pseudolines obtained by such extensions are equivalent to *pseudoconfigurations of points*, and can be considered as oriented matroids of rank 3 (cf. Chapter 5.3 of [BLW⁺99]).

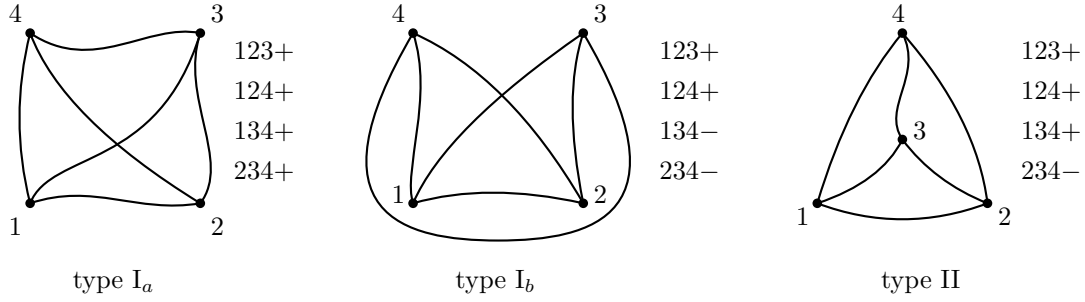


Figure 3: The three types of topological drawings of K_4 in the plane.

2.1 Generalized Signotopes

Let D be a topological drawing of a complete graph in the plane. Assign an *orientation* $\chi(abc) \in \{+, -\}$ to each ordered triple abc of vertices. The sign $\chi(abc)$ indicates whether we go counterclockwise or clockwise around the triangle if we traverse the edges $(a, b), (b, c), (c, a)$ in this order.

If D is a straight-line drawing of K_n , then the underlying point set $S = \{s_1, \dots, s_n\}$ has to be in general position (no three points lie on a line). Assuming that the points are sorted from left to right, then for every 4-tuple s_i, s_j, s_k, s_l with $i < j < k < l$ the sequence $\chi(ijk), \chi(ijl), \chi(ikl), \chi(jkl)$ (index-triples in lexicographic order) is *monotone*, i.e., there is at most one sign-change. A *signotope* is a mapping $\chi : \binom{[n]}{3} \rightarrow \{+, -\}$ with the above monotonicity property. Signotopes are in bijection with Euclidean pseudoline arrangements [FW01] and can be used to characterize pseudolinear drawings [BFK15, Theorem 3.2].

Let us now consider topological drawings of the complete graph. There are two types of drawings of K_4 on the sphere: type I has one crossing and type II has no crossing. Type I can be drawn in two different ways in the plane: in type I_a the crossing is only incident to bounded faces and in type I_b the crossing lies on the outer face; see Figure 3.

A drawing of K_4 with vertices a, b, c, d can be characterized in terms of the sequence of orientations $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$. The drawing is

- ▶ of type I_a or type I_b iff the sequence is $++++, ++--, +- -+, - + +- , - - ++, or - - - -$; and
- ▶ of type II iff the number of $+$'s (and $-$'s respectively) in the sequence is odd.

Therefore there are at most two sign-changes in the sequence $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$ and, moreover, any such sequence is in fact induced by a topological drawing of K_4 . Allowing up to two sign-changes is equivalent to forbidding the two patterns $+ - + -$ and $- + - +$.

A mapping $\chi : [n]_3 \rightarrow \{+, -\}$ is called *alternating* if $\chi(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}) = \text{sgn}(\sigma) \cdot \chi(i_1, i_2, i_3)$ for any distinct indices i_1, i_2, i_3 and any permutation $\sigma \in S_3$. Here we denoted by $[n]_3$ the set of all triples (a, b, c) with $a, b, c \in [n]$ pairwise distinct. If χ is alternating and avoids the two patterns $+ - + -$ and $- + - +$ on sorted indices, i.e., $\chi(ijk), \chi(ijl), \chi(ikl), \chi(jkl)$ has at most two sign-changes for all $i < j < k < l$, then it avoids the two patterns in $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$ for any pairwise distinct $a, b, c, d \in [n]$. We refer to this as the *symmetry property* of the forbidden patterns.

The symmetry property allows us to define *generalized signotopes* as alternating mappings $\chi : [n]_3 \rightarrow \{+, -\}$ with at most two sign-changes on $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$ for any pairwise different $a, b, c, d \in [n]$. We conclude:

Proposition 1. *Every topological drawing of K_n induces a generalized signotope on n elements.*

3 Kirchberger's Theorem

Two closed sets $A, B \subseteq \mathbb{R}^d$ are called *separable* if there exists a hyperplane H separating them, i.e., $A \subset H_1$ and $B \subset H_2$ with H_1, H_2 being the two closed half-spaces defined by H . It is well-known that, if two non-empty compact sets A, B are separable, then they can also be separated by a hyperplane H containing points of A and B . *Kirchberger's Theorem* (see [Kir03] or [Bar02]) asserts that two finite point sets $A, B \subseteq \mathbb{R}^d$ are separable if and only if for every $C \subseteq A \cup B$ with $|C| = d + 2$, $C \cap A$ and $C \cap B$ are separable.

Goodman and Pollack [GP82] proved duals of Kirchberger's Theorem and further theorems like Radon's, Helly's, and Carathéodory's Theorem for arrangements of pseudolines. Their results also transfer to pseudoconfigurations of points and thus to pseudolinear drawings. To be more precise, they proved a natural generalization of Kirchberger's Theorem to pseudoline-arrangements in the plane which, by duality, is equivalent to a separating statement on pseudoconfigurations of points in the plane (cf. Theorem 4.8 and Remark 5.2 in [GP82]).

The 2-dimensional version of Kirchberger's Theorem can be formulated in terms of triple orientations. We show a generalization for topological drawings using generalized signotopes. Two sets $A, B \subseteq [n]$ are *separable* if there exist $i, j \in A \cup B$ such that $\chi(i, j, x) = +$ for all $x \in A \setminus \{i, j\}$ and $\chi(i, j, x) = -$ for all $x \in B \setminus \{i, j\}$. In this case we say that ij *separates* A from B and write $\chi(i, j, A) = +$ and $\chi(i, j, B) = -$. Moreover, if we can find $i \in A$ and $j \in B$, we say that A and B are *strongly separable*. As an example, consider the 4-element generalized signotope of the type I_b drawing of K_4 from Figure 3. The sets $A = \{1, 2\}$ and $B = \{3, 4\}$ are strongly separable with $i = 2$ and $j = 3$ because $\chi(2, 3, 1) = +$ and $\chi(2, 3, 4) = -$.

Theorem 1 (Kirchberger's Theorem for Generalized Signotopes). *Let $\chi : [n]_3 \rightarrow \{+, -\}$ be a generalized signotope, and let $A, B \subseteq [n]$ be two non-empty sets. If for every $C \subseteq A \cup B$ with $|C| = 4$, the sets $A \cap C$ and $B \cap C$ are separable, then A and B are strongly separable.*

Note that, since every topological drawing yields a generalized signotope, Theorem 1 yields a way of generalizing Kirchberger's Theorem to topological drawings of complete graphs. We remark that also the reverse of the theorem is true: If A and B are separable, then for every $C \subseteq A \cup B$ with $|C| = 4$, the sets $A \cap C$ and $B \cap C$ are separable.

Proof. First we prove that all 4-tuples $C \subseteq A \cup B$ with $C \cap A$ and $C \cap B$ non-empty which are separable are also strongly separable. This can be verified looking at Tables 1 and 2, which show that, in every weakly-separable generalized signotopes on $\{a, b_1, b_2, b_3\}$ and $\{a_1, a_2, b_1, b_2\}$, respectively, there is a strong separator of the sets $\{a\}$ and $\{b_1, b_2, b_3\}$ or $\{a_1, a_2\}$ and $\{b_1, b_2\}$, respectively. Recall that every set is strongly separable from the empty set. Hence in the following we assume that all such 4-tuples from $A \cup B$ are strongly separable.

By symmetry we may assume $|A| \leq |B|$. First we consider the cases $|A| = 1, 2, 3$ individually and then the case $|A| \geq 4$.

Let $A = \{a\}$ and let B' be a maximal subset of B such that B' is strongly separated from $\{a\}$, and let $b \in B'$ be such that $\chi(a, b, B') = -$. Suppose that $B' \neq B$, then there is a $b^* \in B \setminus B'$ with

$$\chi(a, b, b^*) = +. \tag{1}$$

$\chi(a, b_1, b_2)$	$\chi(a, b_1, b_3)$	$\chi(a, b_2, b_3)$	$\chi(b_1, b_2, b_3)$	list of separators
+	+	+	+	<u>ab₃</u> , <u>b₁a</u> , <u>b₁b₃</u>
+	+	+	-	<u>ab₃</u> , <u>b₁a</u> , <u>b₁b₂</u> , <u>b₂b₃</u>
+	+	-	+	<u>ab₂</u> , <u>b₁a</u> , <u>b₁b₃</u> , <u>b₃b₂</u>
+	+	-	-	<u>ab₂</u> , <u>b₁a</u> , <u>b₁b₂</u>
+	-	+	+	(no separators)
+	-	-	+	<u>ab₂</u> , <u>b₃a</u> , <u>b₃b₂</u>
+	-	-	-	<u>ab₂</u> , <u>b₁b₂</u> , <u>b₃a</u> , <u>b₃b₁</u>
-	+	+	+	<u>ab₃</u> , <u>b₁b₃</u> , <u>b₂a</u> , <u>b₂b₁</u>
-	+	+	-	<u>ab₃</u> , <u>b₂a</u> , <u>b₂b₃</u>
-	+	-	-	(no separators)
-	-	+	+	<u>ab₁</u> , <u>b₂a</u> , <u>b₂b₁</u>
-	-	+	-	<u>ab₁</u> , <u>b₂a</u> , <u>b₂b₃</u> , <u>b₃b₁</u>
-	-	-	+	<u>ab₁</u> , <u>b₂b₁</u> , <u>b₃a</u> , <u>b₃b₂</u>
-	-	-	-	<u>ab₁</u> , <u>b₃a</u> , <u>b₃b₁</u>

Table 1: Separators for generalized signotopes on $\{a, b_1, b_2, b_3\}$. Strong separators are underlined.

By maximality of B' we cannot use the pair a, b^* for a strong separation of $\{a\}$ and $B' \cup \{b^*\}$. Hence, for some $b' \in B'$:

$$\chi(a, b^*, b') = +. \quad (2)$$

Since χ is alternating (1) and (2) together imply $b' \neq b$. Since $b' \in B'$ we have $\chi(a, b, b') = -$. From this together with (1) and (2) it follows that the four element set $\{a, b, b', b^*\}$ has no separator. This is a contradiction, whence $B' = B$.

As a consequence we obtain:

- Every one-element set $\{a\}$ with $a \in A$ can be strongly separated from B . Since χ is alternating there is a unique $b(a) \in B$ such that $\chi(a, b(a), B) = -$.

Now we look at the case where $A = \{a_1, a_2\}$. Let $b_i = b(a_i)$, i.e., $\chi(a_i, b_i, B) = -$ for $i = 1, 2$. If $\chi(a_1, b_1, a_2) = +$ or $\chi(a_2, b_2, a_1) = +$, then a_1b_1 or a_2b_2 , respectively, is a strong separator for A and B . Therefore, we may assume that $\chi(a_1, b_1, a_2) = -$, $\chi(a_2, b_2, a_1) = -$ and therefore $b_1 \neq b_2$. We get the sequence $+- -+$ for the four element set $\{a_1, a_2, b_1, b_2\}$ which has no strong separator, a contradiction.

Let $A = \{a_1, a_2, a_3\}$. Suppose that A is not separable from B . Let $b_i = b(a_i)$, i.e., $\chi(a_i, b_i, B) = -$ for $i = 1, 2, 3$. For $i, j \in \{1, 2, 3\}$, $i \neq j$ we define $s_{ij} = \chi(a_i, b_i, a_j)$.

If $s_{ij} = +$ for some i and all $j \neq i$, then a_ib_i separates A from B . Hence, for each i there exists j with $s_{ij} = -$.

If $s_{ij} = s_{ji} = -$, then since χ is alternating $b_i \neq b_j$ and $\{a_i, a_j, b_i, b_j\}$ corresponds to the row $+- -+$ in Table 2, i.e., there is no strong separator. Hence, at least one of s_{ij} and s_{ji} is $+$.

These two conditions imply that we can relabel the elements of A such that $s_{12} = s_{23} = s_{31} = +$ and $s_{13} = s_{21} = s_{32} = -$. Suppose that $b_i = b_j = b$ for some $i \neq j \in \{1, 2, 3\}$, then the four elements $\{b, a_1, a_2, a_3\}$ have the pattern $-+ -*$. Avoiding the forbidden pattern, we get $-+ --$ in Table 1, i.e., there is no strong separator. This contradiction shows that b_1, b_2, b_3 must be pairwise distinct.

$\chi(a_1a_2b_1)$	$\chi(a_1a_2b_2)$	$\chi(a_1b_1b_2)$	$\chi(a_2b_1b_2)$	list of separators
+	+	+	+	<u>$a_2a_1, a_2b_2, b_1a_1, b_1b_2$</u>
+	+	+	-	<u>a_2a_1, a_2b_1, b_1a_1</u>
+	+	-	+	<u>a_2a_1, a_2b_2, b_2a_1</u>
+	+	-	-	<u>$a_2a_1, a_2b_1, b_2a_1, b_2b_1$</u>
+	-	+	+	<u>a_1b_2, b_1a_1, b_1b_2</u>
+	-	-	+	(no separators)
+	-	-	-	<u>a_2b_1, b_2a_2, b_2b_1</u>
-	+	+	+	<u>a_2b_2, b_1a_2, b_1b_2</u>
-	+	+	-	(no separators)
-	+	-	-	<u>a_1b_1, b_2a_1, b_2b_1</u>
-	-	+	+	<u>$a_1a_2, a_1b_2, b_1a_2, b_1b_2$</u>
-	-	+	-	<u>a_1a_2, a_1b_2, b_2a_2</u>
-	-	-	+	<u>a_1a_2, a_1b_1, b_1a_2</u>
-	-	-	-	<u>$a_1a_2, a_1b_1, b_2a_2, b_2b_1$</u>

Table 2: Separators for generalized signotopes on $\{a_1, a_2, b_1, b_2\}$. Strong separators are underlined.

From $s_{32} = -$ and $s_{31} = +$ we find that $\{b_3, a_1, a_2, a_3\}$ corresponds to a row of type $* + -*$ in Table 1. We conclude that the strong separator of $\{b_3, a_1, a_2, a_3\}$ is a_2b_3 . In particular,

$$\chi(b_3, a_1, a_2) = +. \quad (3)$$

Now consider $\{a_1, a_2, b_1, b_3\}$. From $s_{12} = +$, equation (3), and $\chi(a_1, b_1, b_3) = -$ we obtain the pattern $- + -*$. Since $- + - +$ is forbidden we obtain

$$\chi(a_2, b_1, b_3) = -. \quad (4)$$

The set $\{a_2, a_3, b_1, b_3\}$ needs a strong separator. The candidate pair a_3b_1 is made impossible by $\chi(a_3, b_1, b_3) = +$, a_3b_3 is made impossible by $s_{32} = -$, and a_2b_3 is made impossible by (4). Hence a_2b_1 is the strong separator and, in particular, it holds

$$\chi(a_2, b_1, a_3) = +. \quad (5)$$

But now the set $\{a_1, a_2, a_3, b_1\}$ has no strong separator. The candidate pair a_1b_1 is impossible because of $s_{13} = -$, a_2b_1 does not separate because $s_{12} = +$, and (5) shows that a_3b_1 cannot separate the set. This contradiction proves the case $|A| = 3$.

For the remaining case $|A| \geq 4$ consider a counterexample (χ, A, B) minimizing the size of the smaller of the two sets. We have $4 \leq |A| \leq |B|$.

Let $a^* \in A$. By minimality $A' = A \setminus \{a^*\}$ is separable from B . Let $a \in A'$ and $b \in B$ such that $\chi(a, b, A') = +$ and $\chi(a, b, B) = -$. Hence

$$\chi(a, b, a^*) = -. \quad (6)$$

Let $b^* = b(a^*)$, i.e., $\chi(a^*, b^*, B) = -$. There is some $a' \in A'$ such that

$$\chi(a^*, b^*, a') = -. \quad (7)$$

If $a' = a$, then $b \neq b^*$ because of (6) and (7). From (6), (7), $\chi(a, b, B) = -$, and $\chi(a^*, b^*, B) = -$ it follows that the four element set $\{a, a^*, b, b^*\}$ has the sign pattern $+ - - +$, hence there is no separation. This shows that $a' \neq a$.

Let $b' = b(a')$. If $b \neq b'$ we look at the four elements $\{a, b, a', b'\}$. It corresponds to $+ - * -$ so that we can conclude $\chi(a, a', b') = -$. If $b = b'$, then $a' \in A'$ implies $\chi(a, b, a') = +$ which yields $\chi(a', b', a) = -$.

Hence, regardless whether $b = b'$ or $b \neq b'$ we have

$$\chi(a', b', a) = - . \tag{8}$$

Since $|A| \geq 4$, we know by the minimality of (χ, A, B) that the set $\{a, b, a', b', a^*, b^*\}$, which has 3 elements of A and at least 1 element of B , is separable. It follows from $\chi(a, b, B) = \chi(a', b', B) = \chi(a^*, b^*, B) = -$ that the only possible strong separators are ab , $a'b'$, and a^*b^* . They, however, do not separate because of (6), (7) and (8) respectively. This contradiction shows that there is no counterexample. \square

4 Carathéodory's Theorem

Carathéodory's Theorem asserts that, if a point x lies in the convex hull of a point set P in \mathbb{R}^d , then x lies in the convex hull of at most $d + 1$ points of P .

As already mentioned in Section 3, Goodman and Pollack [GP82] proved a dual of Carathéodory's Theorem, which transfers to pseudolinear drawings.

A more general version of Carathéodory's Theorem in the plane is due to Balko, Fulek, and Kynčl, who provided a generalization to topological drawings. In this section, we present a shorter proof for their theorem.

Theorem 2 (Carathéodory for Topological Drawings [BFK15, Lemma 4.7]). *Let D be a topological drawing of K_n and let $x \in \mathbb{R}^2$ be a point contained in a bounded connected component of $\mathbb{R}^2 - D$. Then there is a triangle which contains x in its interior.*

Proof. Suppose towards a contradiction that there is a pair (D, x) violating the claim. We choose D minimal with respect to the number of vertices n .

Let a be a vertex of the drawing. If we remove all incident edges of a from D , then by minimality of the example x becomes a point of the outer face. Therefore, if we consecutively remove the incident edges of a one by one, we find a last subdrawing D' such that x is still in a bounded face. Let ab be the edge such that in the drawing $D' - ab$ the point x is in the outer face.

There is a simple path P connecting x to infinity, which does not cross any of the edges in $D' - ab$. By the choice of D' , P has at least one crossing with ab . We choose P minimal with respect to the number of crossings with ab .

We claim that P intersects ab exactly once. Suppose that P crosses ab more than once. Then there is a *lense* C formed by P and ab , that is, two crossings of P and ab such that a simple closed curve ∂C , composed of the subpath P_1 of P and the subpath P_2 of ab between the crossings, encloses a simply connected region C , see Figure 4(a).

Now consider the path P' from x to infinity which is obtained from P by replacing the subpath P_1 by a path P'_2 which is a close copy of P_2 in the sense that it has the same crossing pattern with all edges in D and the same topological properties, but is disjoint from ab . As P was chosen minimal with respect to the number of crossings with ab , there has to be an edge of the drawing D' that intersects P'_2 (and by the choice of P'_2 also P_2). This edge can cross neither P , by construction, nor ab again, so it has one of its endpoints inside the lense C and one outside C . We can therefore choose one of these endpoints, say c , of that edge such that the edge bc in D' must intersect ∂C . But since they are adjacent, bc cannot intersect ab and by

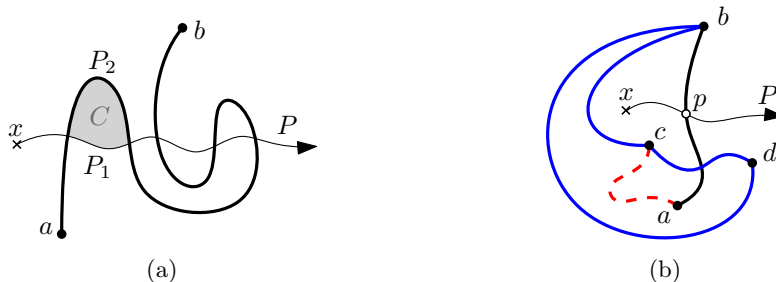


Figure 4: (a) and (b) give an illustration of the proof of Theorem 2.

the choice of P it does not intersect P . The contradiction shows that P crosses ab in a unique point p .

If a has another neighbor c in the drawing D' then, since only edges incident to a have been removed there is an edge connecting b to c in D' . The edges ac and bc do not cross P , so x is in the interior of the triangle abc and we are done.

If there is no edge ac in D' , then $\deg(a) = 1$ in D' . As x is not in the outer face of D' , there must be an edge cd in D' which intersects the partial segment of the edge ab starting in a and ending in p , in its interior, let c be the point on the same side of ab as x ; see Figure 4(b). The edges bc and bd of D' cross neither P nor ab . Consequently, the triangle bcd (drawn blue) must contain a in its interior. We claim that the edge ac in the original drawing D (drawn red dashed) lies completely inside the triangle bcd : The bounded region defined by the edges ab , cd , and bd of D' contains a and c . Since D is a topological drawing ac has no crossing with ab , and cd , therefore, it has no crossing with bd . This proves the claim. Now the path P does not intersect ac , and the only edge of the triangle abc intersected by P is ab . Therefore, x lies in the interior of the triangle abc . This contradicts the assumption that (D, x) is a counterexample. \square

4.1 Colorful Carathéodory Theorem

Bárány [Bár82] generalized Carathéodory's Theorem as follows: If P_0, \dots, P_d are finite point sets from \mathbb{R}^d whose convex hulls have a common point $x \in \text{conv}(P_0) \cap \dots \cap \text{conv}(P_d)$, then x lies in a colorful simplex spanned by $p_0 \in P_0, \dots, p_d \in P_d$. This result is widely known as the *Colorful Carathéodory Theorem*. A strengthening, known as the *Strong Colorful Carathéodory Theorem*, was shown by Holmsen, Pach, and Tverberg [HPT08] (cf. [Kal09]): It is sufficient if there is a point x with $x \in \text{conv}(P_i \cup P_j)$ for any $i \neq j$, to find a colorful simplex. The Strong Colorful Carathéodory Theorem was further generalized to oriented matroids by Holmsen [Hol16]. In particular, the theorem applies to pseudolinear drawings (which are in correspondence with oriented matroids of rank 3).

There are several ways to prove Colorful Carathéodory Theorem for pseudolinear drawings. Besides Holmsen's proof [Hol16], which uses sophisticated methods from topology, we have also convinced ourself that Bárány's proof [Bár82] can be adapted to pseudoconfigurations of points in the plane. However, Bárány's proof idea does not directly generalize to higher dimensions because oriented matroids of higher ranks do not necessarily have a representation in terms of pseudoconfigurations of points in d -space (cf. [BLW⁺99, Chapter 1.4]).

Another way to prove the Strong Colorful Carathéodory Theorem for pseudolinear drawings is by computer assistance: By using the order type database (cf. [AAK02] and [SSS20, Section 6.1]) or by using a SAT model one can verify that no pseudoconfigurations of 10 points violates the conditions.

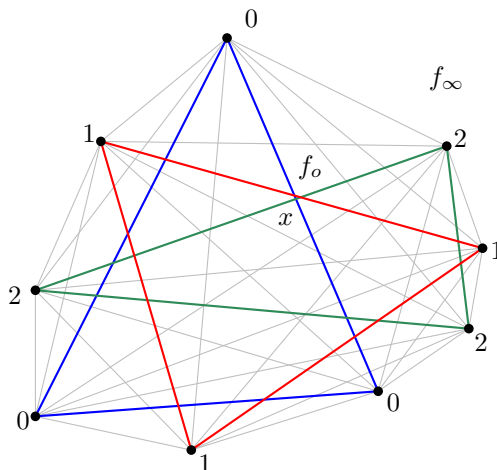


Figure 5: A face-convex drawings of K_9 . If the cell f_o is chosen as the outer face, then the conditions of the Colorful Carathéodory’s Theorem are violated. The special cell of the pseudolinear drawing is marked by f_∞ .

The following result shows that in the convexity hierarchy of topological drawings of K_n the Colorful Carathéodory Theorem is not valid beyond the class of pseudolinear drawings.

Proposition 2. *The Colorful Carathéodory Theorem does not hold for the face-convex drawing of Figure 5.*

Proof. The drawing depicted in Figure 5 is face-convex because it is obtained from a straight-line drawing by choosing f_o as outer face. The point x is contained in the three colored triangles. This point is separated from the outer face only by three colored edges. Therefore, there is no triangle containing x with vertices of each of the three colors. \square

5 Helly’s Theorem

The *Helly number* of a family of sets \mathcal{F} with empty intersection is the size of the smallest subfamily of \mathcal{F} with empty intersection. *Helly’s Theorem* asserts that the Helly number of a family of n convex sets S_1, \dots, S_n from \mathbb{R}^d is at most $d + 1$, i.e., the intersection of S_1, \dots, S_n is non-empty if the intersection of every $d + 1$ of these sets is non-empty.

In the following we discuss the Helly number in the context of topological drawings, where each set corresponds to the bounded region of an induced subdrawing.

From the results of Goodman and Pollack [GP82] it follows that Helly’s Theorem generalizes to pseudoconfigurations of points in two dimensions, and thus for pseudolinear drawings. A more general version of Helly’s Theorem for oriented matroids (with the intersection property) was shown by Bachem and Wanka [BW88]. They prove Helly’s and Radon’s Theorem in the context of oriented matroids with the “intersection property”. Since all oriented matroids of rank 3 have the intersection property (cf. [BW88] and [BW89]) and oriented matroids of rank 3 correspond to pseudoconfigurations of points, which in turn yield pseudolinear drawings, the two theorems apply to pseudolinear drawings.

We show that Helly’s Theorem does not hold for face-convex drawings, moreover, the Helly number can be arbitrarily large in face-convex drawings. Note that the following proposition

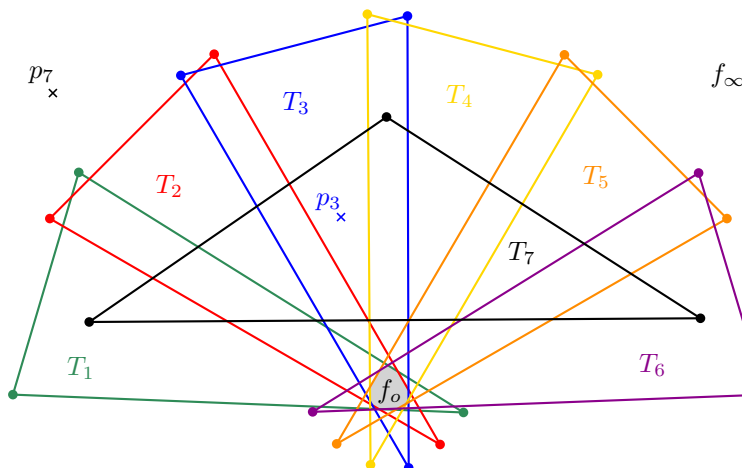


Figure 6: A drawing D of K_{21} is obtained by adding the remaining edges as straight-segments. Making the gray cell f_o the outer face we obtain a face-convex drawing with Helly number 7.

does not contradict the Topological Helly Theorem [Hel30] (cf. [GPP⁺17]) because there are triangles whose intersection is disconnected.

Proposition 3. *Helly’s Theorem does not generalize to face-convex drawings. Moreover, for every integer $n \geq 3$, there exists a face-convex drawing of K_{3n} with Helly number at least n , i.e., there are n triangles such that for any $n - 1$ of the triangles, their bounded sides have a common interior point, but the intersection of the bounded sides of all n triangles is empty.*

Proof. Consider a straight-line drawing D of K_{3n} with n triangles T_i as shown for the case $n = 7$ in Figure 6. With D' we denote the drawing obtained from D by making the gray cell f_o the outer face. Let O_i be the side of T_i , which is bounded in D' . For $1 \leq i < n$ the set O_i corresponds to the outside of T_i in D while O_n corresponds to the inside of T_n .

In D' we have $\bigcap_{i=1}^{n-1} O_i \neq \emptyset$, indeed any point p_n which belongs to the outer face of D is in this intersection. Since $T_n \subset \bigcup_{i=1}^{n-1} T_i$ we have $T_n \cap \bigcap_{i=1}^{n-1} O_i = \emptyset$ i.e., $\bigcap_{i=1}^n O_i = \emptyset$. For each $i \in \{1, \dots, n-1\}$ there is a point $p_i \in T_i \cap T_n$ which is not contained in any other T_j . Therefore, $p_i \in \bigcap_{j=1; j \neq i}^n O_j$.

In summary, the intersection of any $n - 1$ of the n sets O_1, \dots, O_n is non-empty but the intersection of all of them is empty. \square

6 Discussion

We conclude this article with three further classical theorems from Convex Geometry.

Lovász (cf. Bárány [Bár82]) generalized Helly’s Theorem as follows: Let $\mathcal{C}_0, \dots, \mathcal{C}_d$ be families of compact convex sets from \mathbb{R}^d such that for every “colorful” choice $C_0 \in \mathcal{C}_0, \dots, C_d \in \mathcal{C}_d$ the intersection $\text{conv}(C_0) \cap \dots \cap \text{conv}(C_d)$ is non-empty. Then, for some k , the intersection $\bigcap \mathcal{C}_k$ is non-empty. This result is known as the *Colorful Helly Theorem*. Kalai and Meshulam [KM05] presented a topological version of the Colorful Helly Theorem, which, in particular, carries over to pseudolinear drawings. Since Helly’s Theorem does not generalize to face-convex drawings (cf. Proposition 3), neither the Colorful Helly Theorem does.

The (p, q) -Theorem (conjectured by Hadwiger and Debrunner, proved by Alon and Kleitman [AK92], cf. [KST18]) says that for any $p \geq q \geq d + 1$ there is a finite number $c(p, q, d)$ with the following property: If \mathcal{C} is a family of convex sets in \mathbb{R}^d , with the property that among any p of them, there are q that have a common point, then there are $c(p, q, d)$ points that cover all the sets in \mathcal{C} . The case $p = q = d + 1$ is Helly’s Theorem, i.e., $c(d + 1, d + 1, d) = 1$. A (p, q) -Theorem for triangles in topological drawings can be derived from [CHP12, Theorem 4.6]:

Theorem 3. *For $p \geq q \geq 2$, there exists a finite number $\tilde{c}(p, q)$ such that, if \mathcal{T} is a family of triangles of a topological drawing and among any p members of \mathcal{T} there are q that have a common point, then there are $\tilde{c}(p, q)$ points that cover all the triangles of \mathcal{T} .*

Last but not least, we would like to mention *Tverberg’s Theorem*, which asserts that every set V of at least $(d + 1)(r - 1) + 1$ points in \mathbb{R}^d can be partitioned into $V = V_1 \dot{\cup} \dots \dot{\cup} V_r$ such that $\text{conv}(V_1) \cap \dots \cap \text{conv}(V_r)$ is non-empty. A generalization of Tverberg’s Theorem applies to pseudolinear drawings [Rou88] and to drawings of K_{3r-2} if r is a prime-power [Öza87] (cf. [BSS81]). Also a generalization of Birch’s Theorem, a weaker version of Tverberg’s Theorem, was recently proven for topological drawings of complete graphs [FS20]. The general case, however, remains unknown. For a recent survey on generalizations of Tverberg’s Theorem, we refer to [BS18].

In future work [BFS⁺20], we study the structure of generalized signotopes in more detail. We show that the number of generalized signotopes on n elements is of order $2^{\Theta(n^3)}$, and that most of them are not induced by a topological drawing.

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