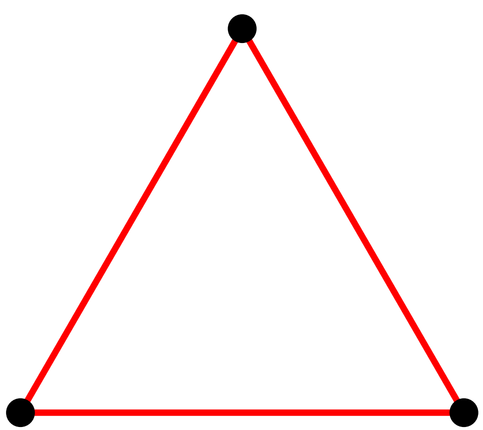
A Note on Universal Point Sets for Planar Graphs

Manfred Scheucher, Hendrik Schrezenmaier, Raphael Steiner

Universal Sets

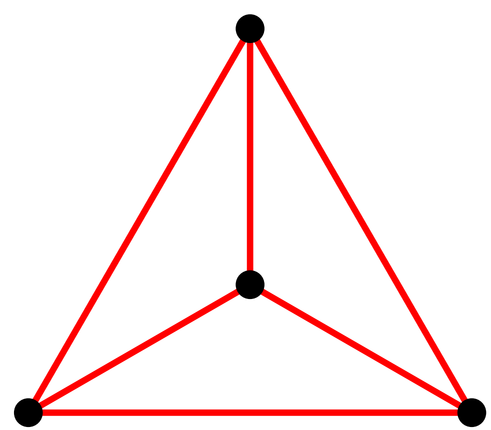
Definition: *n-universal* point set S:
$\forall$ planar n-vertex graph G can be drawn straight-line on S.

$n = 3$:



(unique)

$n = 4$:



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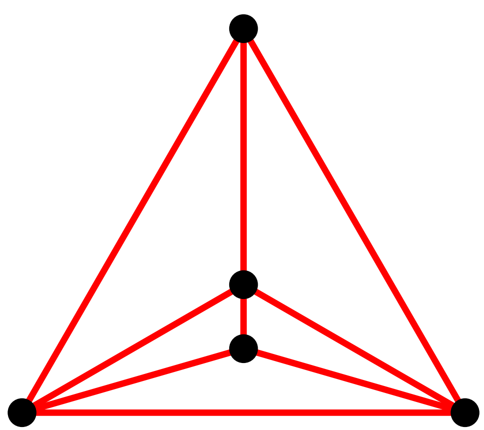
$n = 5$:



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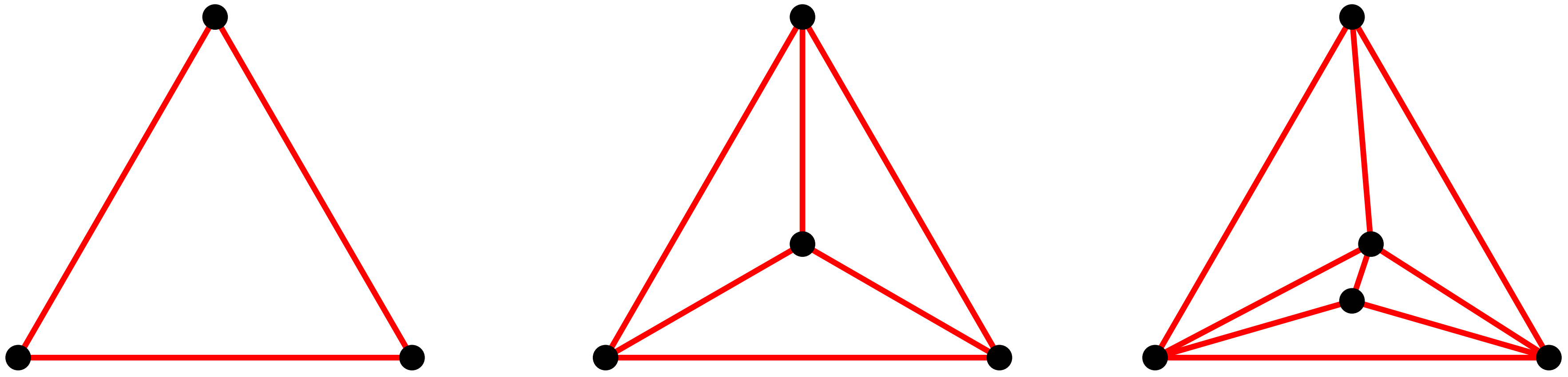

w.l.o.g.: n-universal sets in general position

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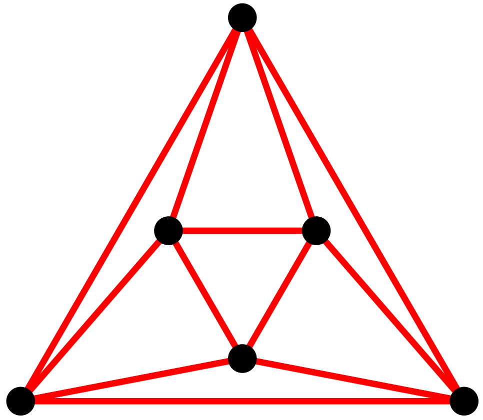
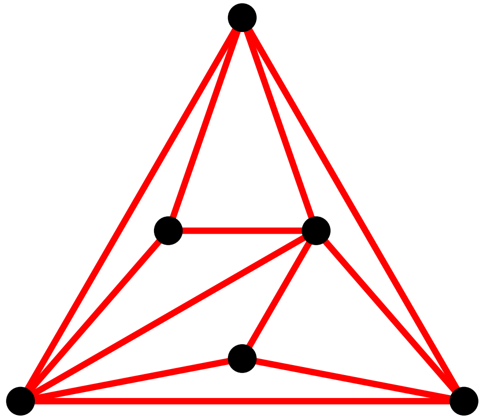
$n = 6$:



degrees: 4-regular



degrees: 3,3,4,4,5,5

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Problem (Brass, Cenek, Duncan, Efrat, Erten, Ismailescu, Kobourov, Lubiw, Mitchell):
What is the smallest size σ of a collection of planar graphs without a simultaneous embedding (conflict collection)?

Upper Bounds

- $(2n-4) \times (n-2)$ grid is n-universal, hence $f(n) = O(n^2)$ [De Fraysseix, Pach, Pollack '90]
- $\ldots$
- $f(n) \leq \frac{n^2}{4} - O(n)$ [Bannister, Cheng, Devanny, Eppstein '14]

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- $\dots$
- $f(n) \leq \frac{n^2}{4} - O(n)$ [Bannister, Cheng, Devanny, Eppstein '14]
- $f_s(n) \leq O(n^{3/2} \log n)$ for stacked triangulations [Fulek and Tóth '15]

Lower Bounds

- Counting arguments
- $f(n) \geq n + \Omega(\sqrt{n})$ [De Fraysseix, Pach, Pollack '90]
- $\ldots$
- $f(n) \geq f_s(n) \geq 1.235n(1 + o(1))$ [Kurowski '04]

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- $f(n) = n$ for $n \leq 10$,
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$|S| \geq 1.293n(1 + o(1))$

$\nexists$ 11-universal set on 11 points

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  $f(n) \geq f_s(n) \geq n + 1$ for $n \geq 15$, and
  $\sigma \leq 7393$ [Cardinal, Hoffmann, Kusters '15]

$\sigma \leq 49$

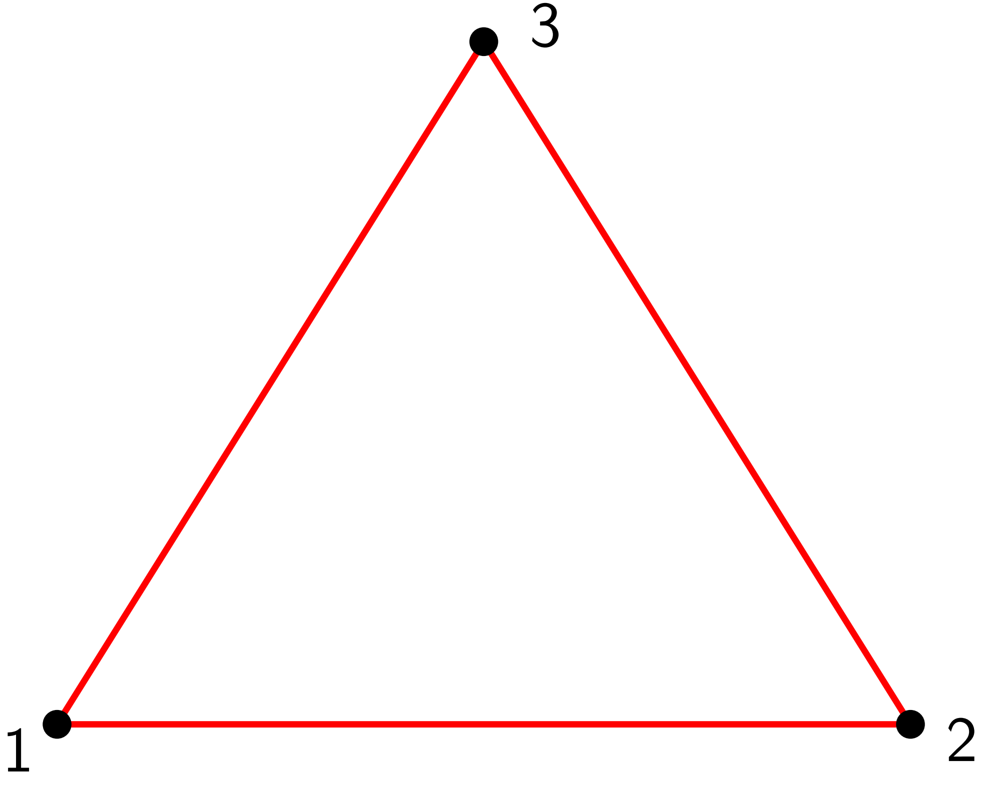
New Lower Bound

Theorem (S., Schrezenmaier, Steiner '19).

$$f_s(n) \geq (1.293 - o(1))n$$

New Lower Bound

Starting from a triangle, a *stacked triangulation* is built up by repeated insertions of degree-3-vertices into triangles.



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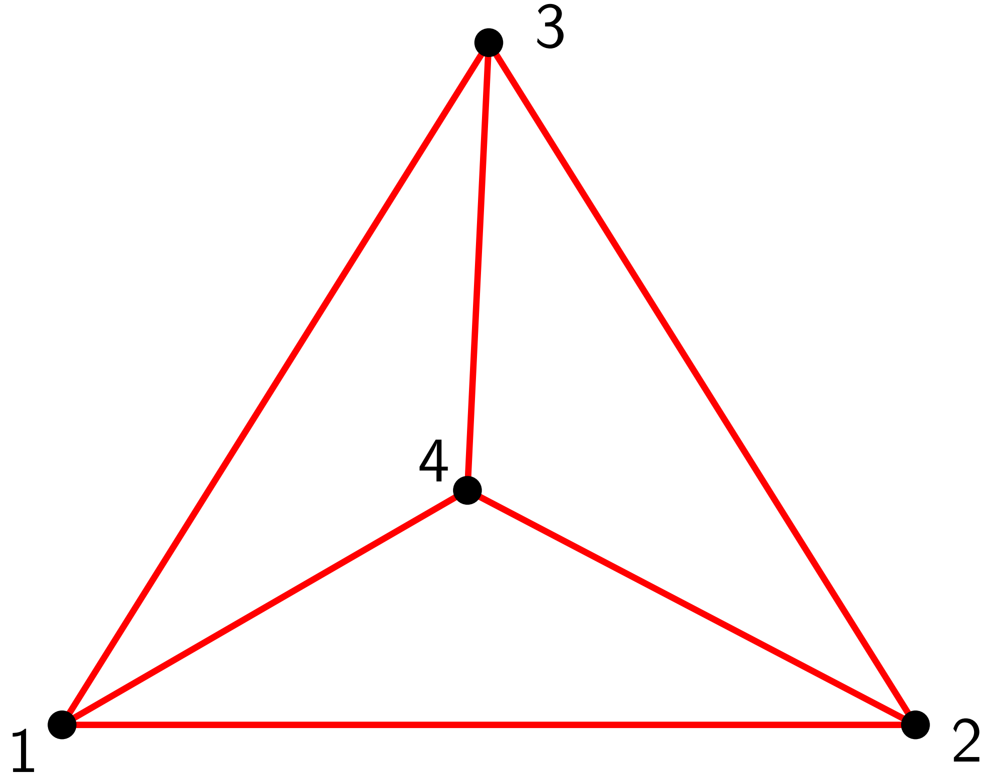

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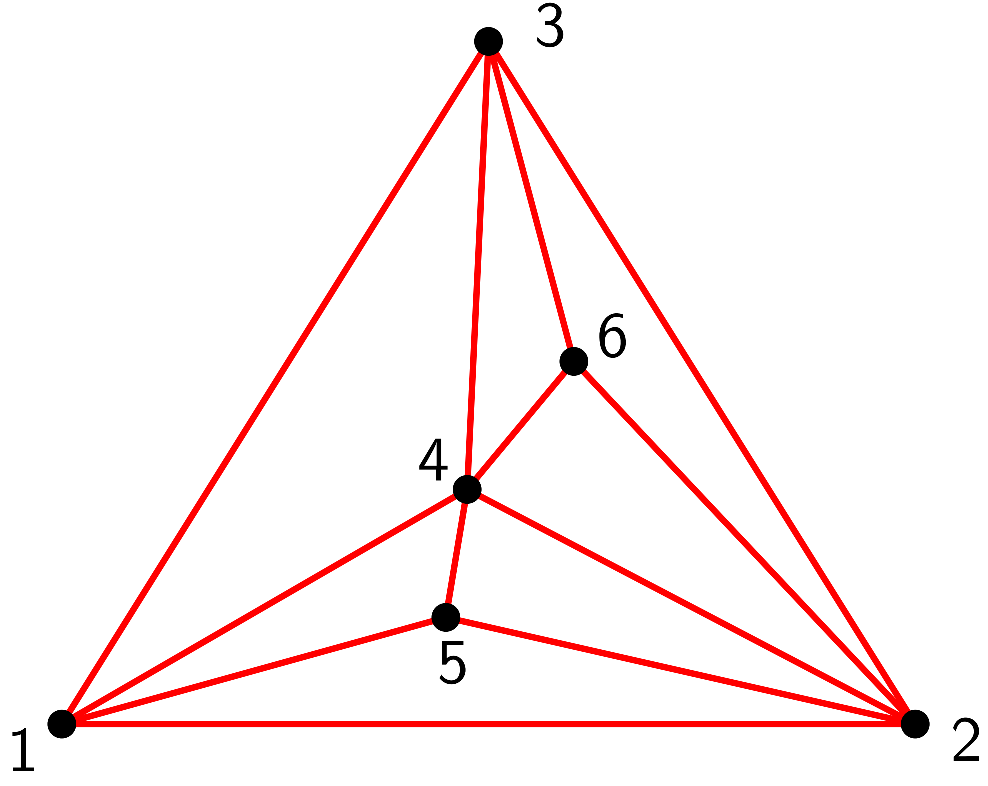

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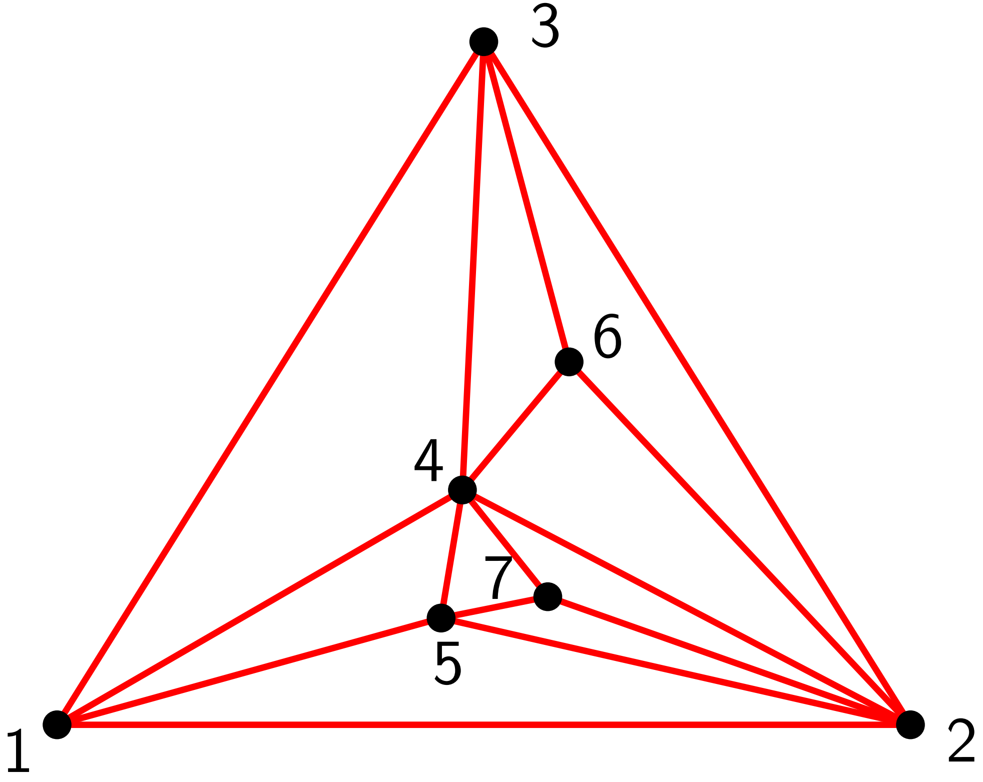

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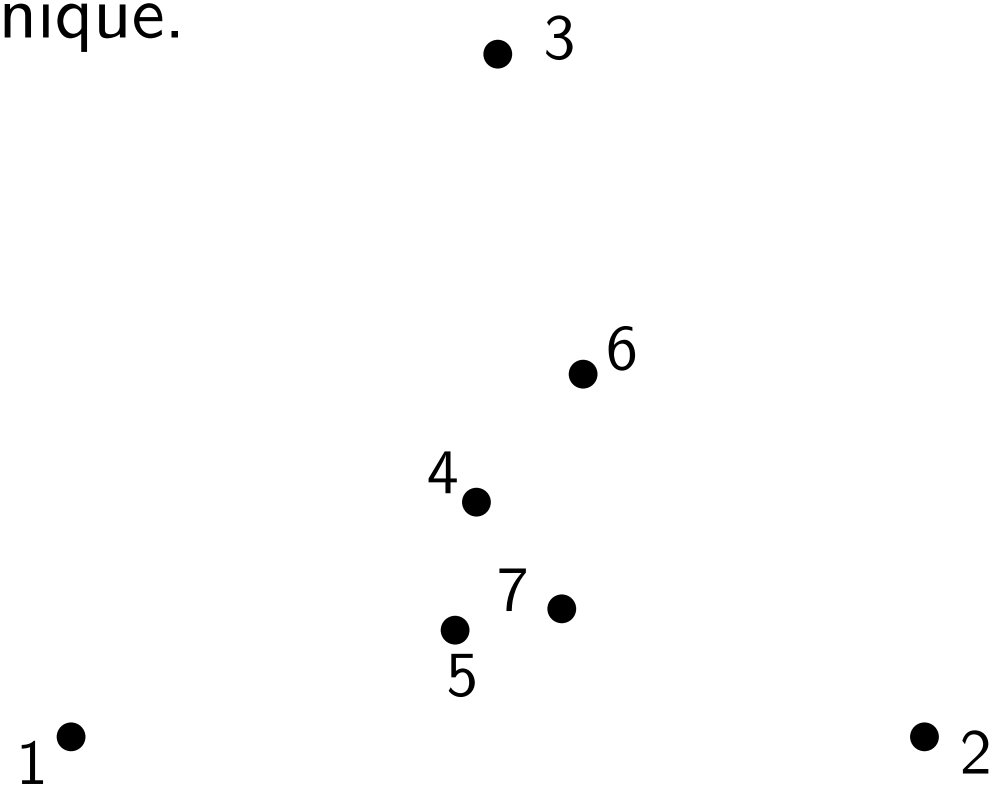
Lemma (Cardinal, Hoffmann, Kusters '15).
The induced labeling is unique.



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Obsv. $\#$ of labeled stacked triangulations: $2^{n-4}(n-3)!$

Corollary. Let m be the size of an n-universal set. Then

$$2^{n-4}(n-3)! \leq \# \text{ labelings of n out of m points} = \frac{m!}{(m-n)!}$$

New Lower Bound

Theorem (S., Schrezenmaier, Steiner '19).

$$f_s(n) \geq (1.293 - o(1))n$$

11-Universal Sets

Theorem (S., Schrezenmaier, Steiner '19).
There is a set of 49 stacked triangulations on 11 vertices without a simultaneous embedding, hence

$$f(11) = f_s(11) = 12 \quad \text{and} \quad \sigma \leq 49.$$

SAT Model

SAT model for a fixed set S and fixed graph $G = (V, E)$:

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- Injective mapping $V \to S$

every vertex v_i has to be mapped:

$$\bigvee_j M_{i,j}$$

no two vertices v_{i_1}, v_{i_2} mapped to the same point:

$$\neg M_{i_1,j} \vee \neg M_{i_2,j}$$

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- No two edges cross

$\forall$ pair of edges (v_1, v_2), (v_3, v_4)
$\forall$ pair of crossing segments (p_1, p_2), (p_3, p_4)

$$\neg M_{v_1,p_1} \vee \neg M_{v_2,p_2} \vee \neg M_{v_3,p_3} \vee \neg M_{v_4,p_4}$$

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. . . but solvers do not terminate . . .

Computer Proof

- Enumerate all triangulations on 11 vertices (1,249)

  via plantri (planar graph generator by Brinkmann and McKay)

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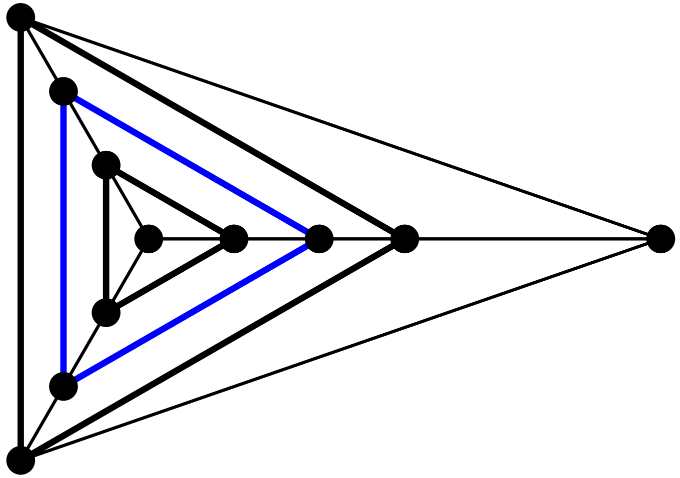
- Enumerate all order types on 11 points (2,343,203,071)

via signotope/chirotope axioms, 20 CPU hours, 100 GB storage

Computer Proof

- Enumerate all triangulations on 11 vertices (1,249)
- Enumerate all order types on 11 points (2,343,203,071)
- Test necessary criterion on point sets



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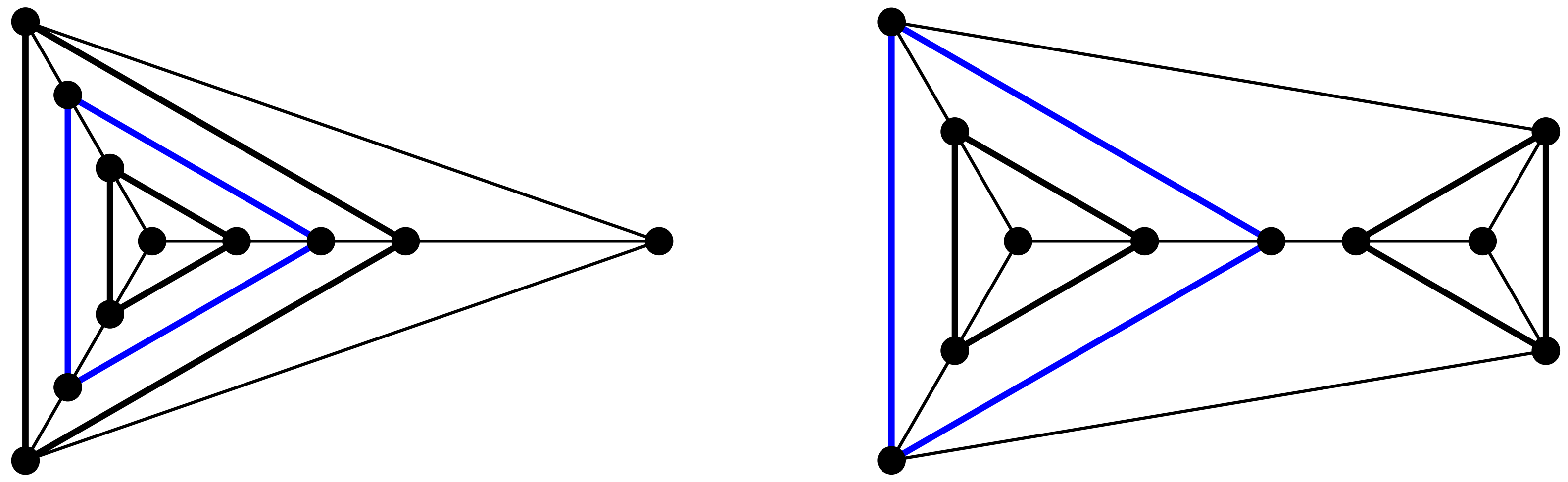

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- Pick $\mathcal{G}$ as set of 11-vertex triangulations with maximum degree 10 and test each pair S and G

via SAT solver, priority queue

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- For remaining $\mathcal{G}$-universal sets, create 0-1-matrix and use IP to find minimal set of triangulations which need to be added (Minimum set cover)
- 500 CPU days later:

previously: 7393 for larger n

conflict collection of 49 stacked triang. on 11 vertices!

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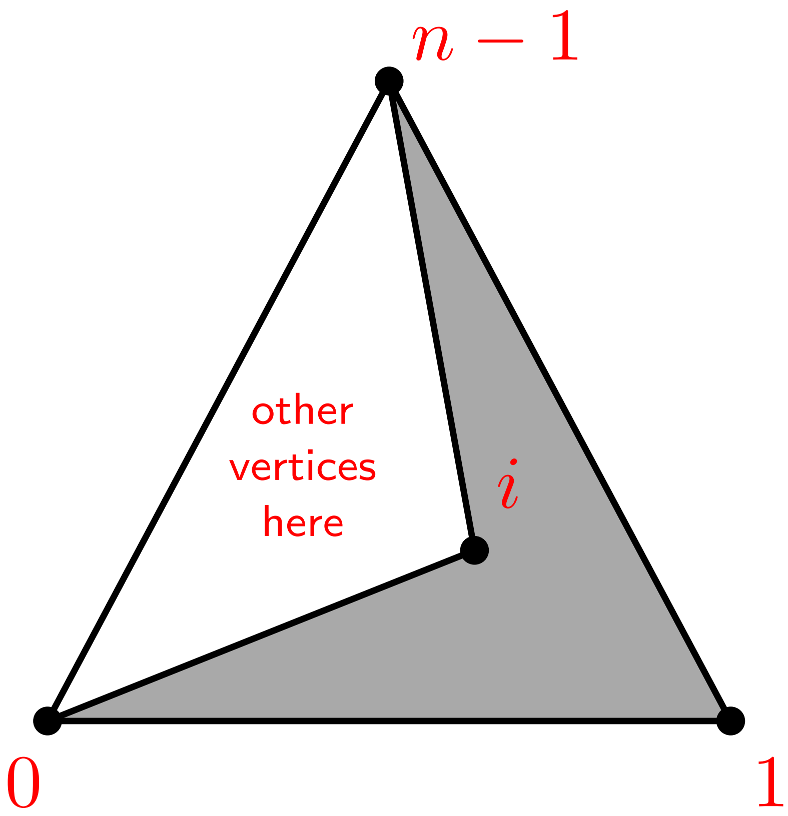

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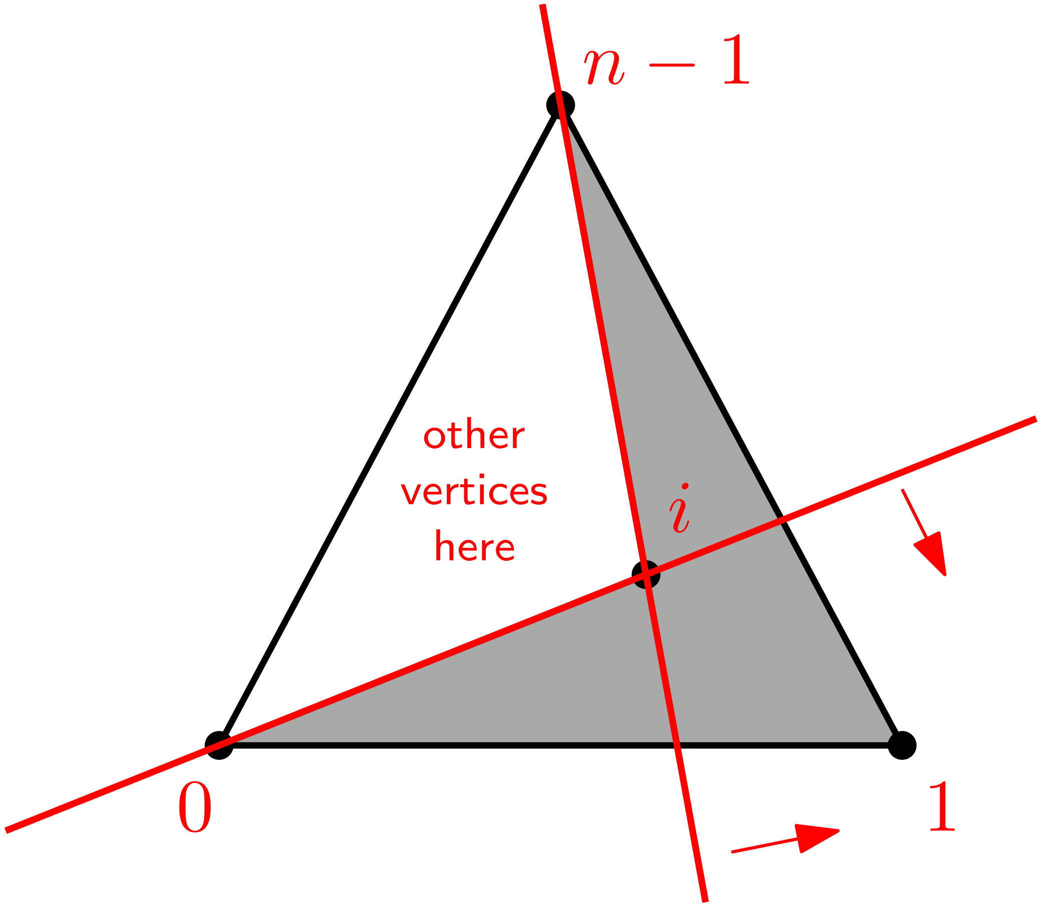

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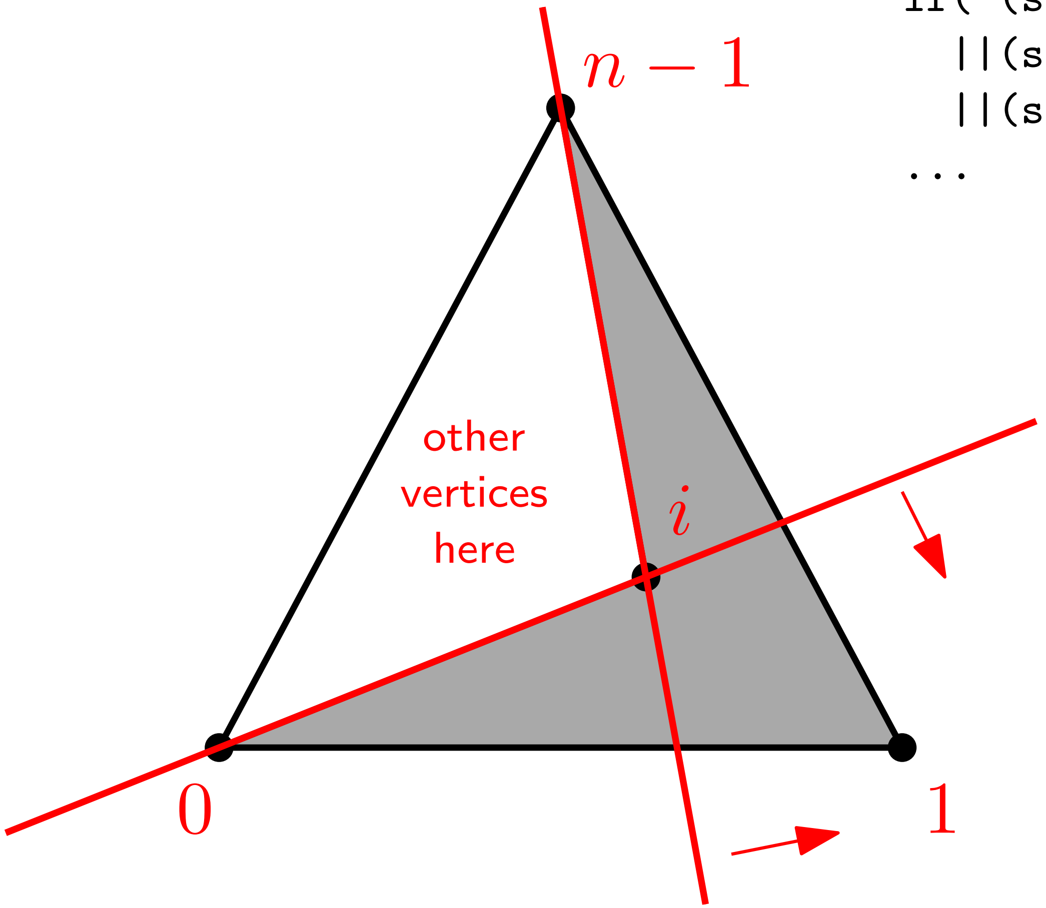


```
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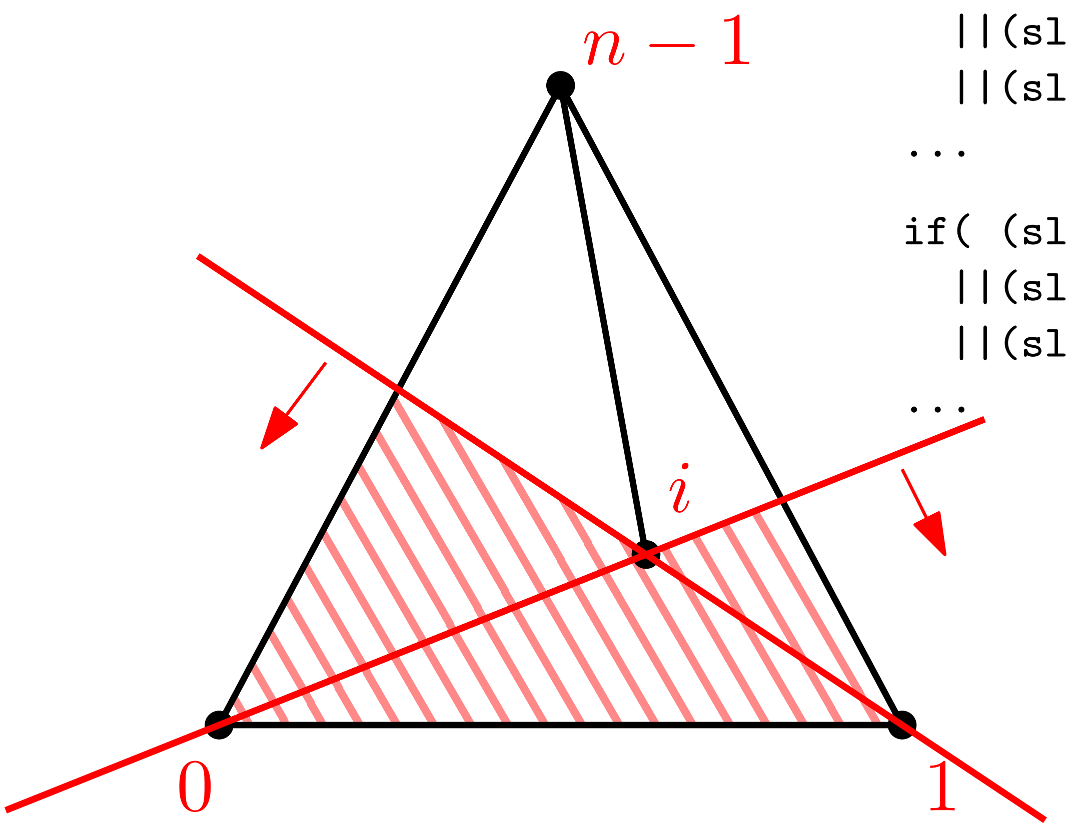


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Thank you for your attention!