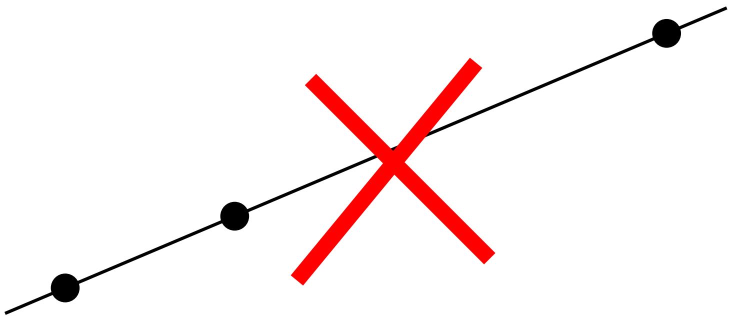
Erdős–Szekeres-type problems in the real projective plane

Martin Balko, Manfred Scheucher, Pavel Valtr

General Position

a finite point set P in the plane is
in general position if $\nexists$ collinear points in P



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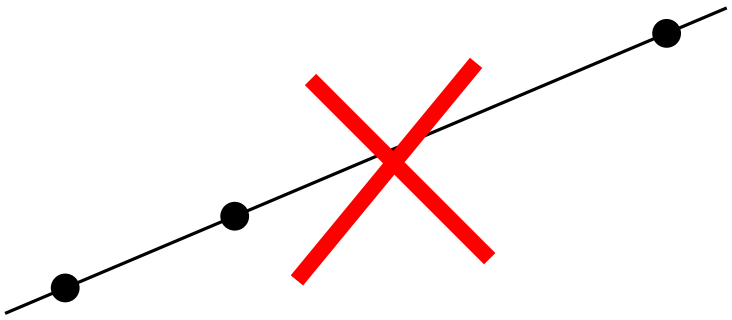
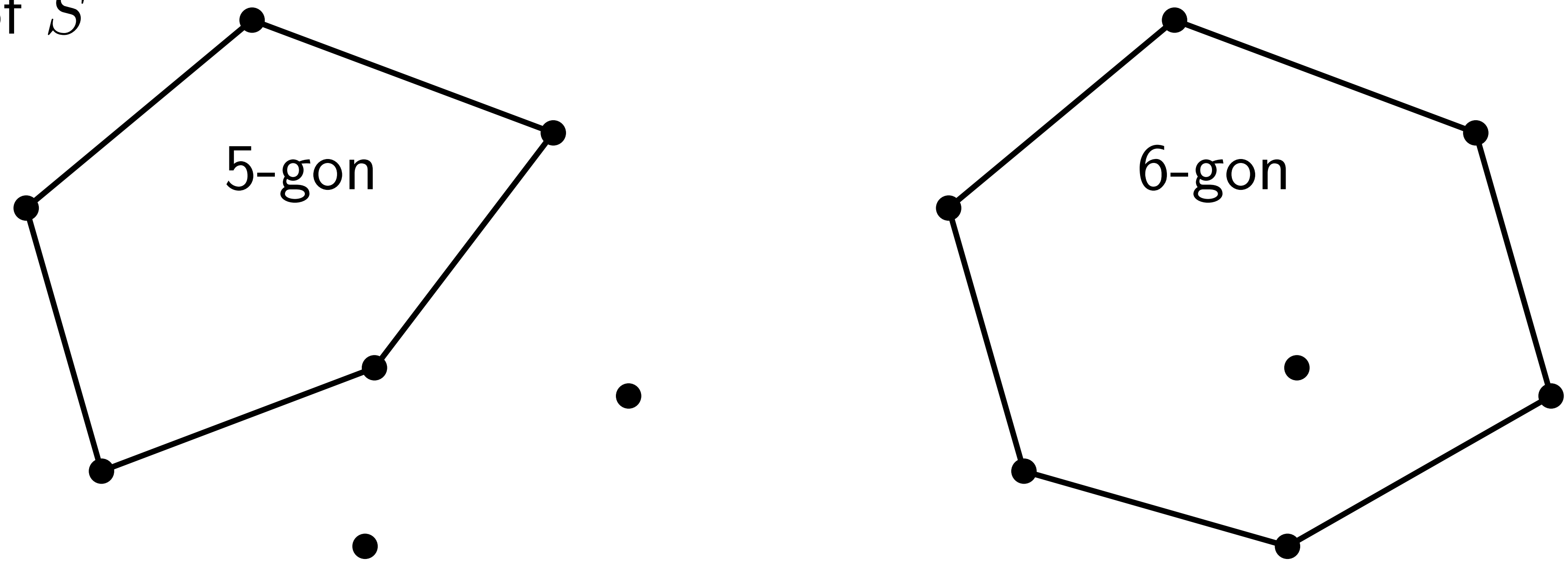


throughout this presentation, every set is in general position

The Affine World

k-Gons

a k-gon (in S) is a convex polygon spanned by k points of S



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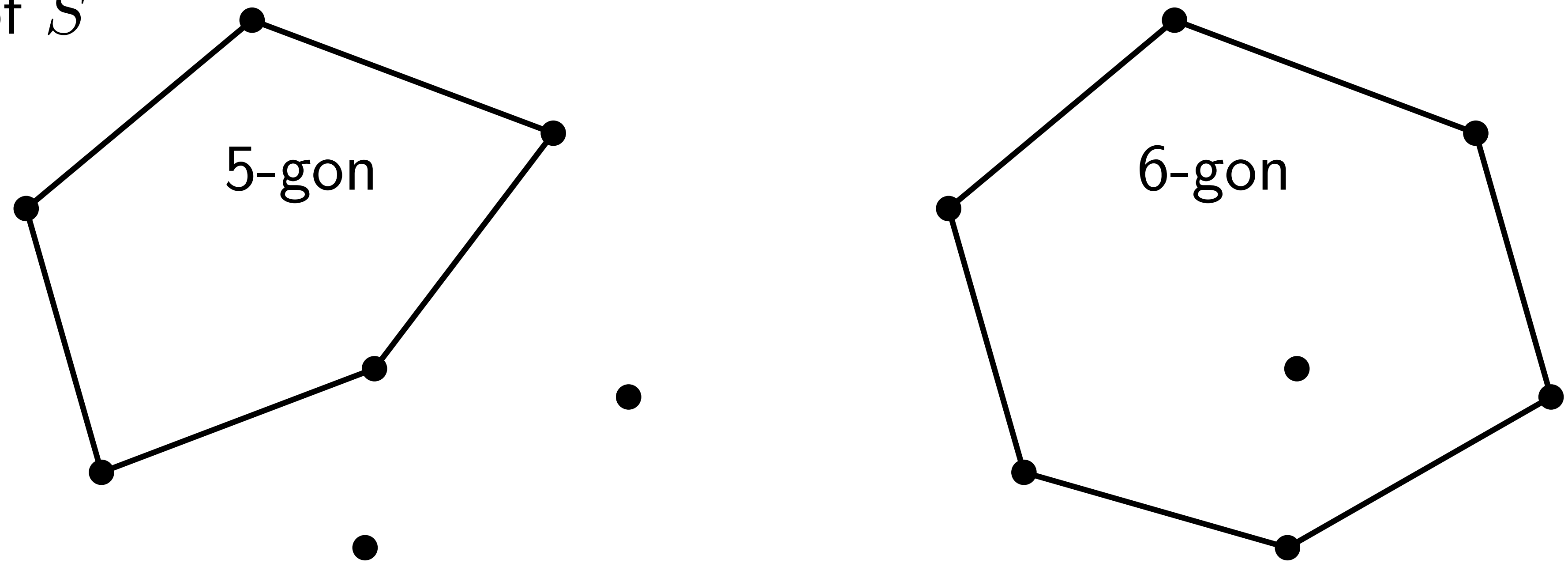

Theorem (Erdős & Szekeres 1935).
$\forall\ k \in \mathbb{N}$, $\exists$ a smallest integer $g(k)$ such that every set of $g(k)$ points determines a k-gon.

k-Gons

Theorem (Erdős & Szekeres '35)

$$2^{k-2} + 1 \leq g(k) \leq \binom{2k-4}{k-2} + 1$$

equality conjectured by Szekeres, Erdős offered 500$ for a proof

k-Gons

Theorem (Erdős & Szekeres '35)
$2^{k-2}+1 \leq g(k) \leq \binom{2k-4}{k-2}+1$

$\vdots$ several improvements of order $4^{k-o(k)}$

Theorem. $g(k) \leq 2^{k+o(k)}$. [Suk '16]

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- $g(k) \leq 2^{k+O(k^{2/3} \log k)}$ [Suk '16]
- $g(k) \leq 2^{k+O(\sqrt{k \log k})}$, also for pseudo-configurations of points [Holmsen, Mojarrad, Pach and Tardos '17]

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Known: $g(4)=5$, $g(5)=9$, $g(6)=17$

↑

computer assisted proof, 1500 CPU hours [Szekeres–Peters '06]

k-Gons

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< 1 hour using SAT solvers [S.'18, Marić '19]

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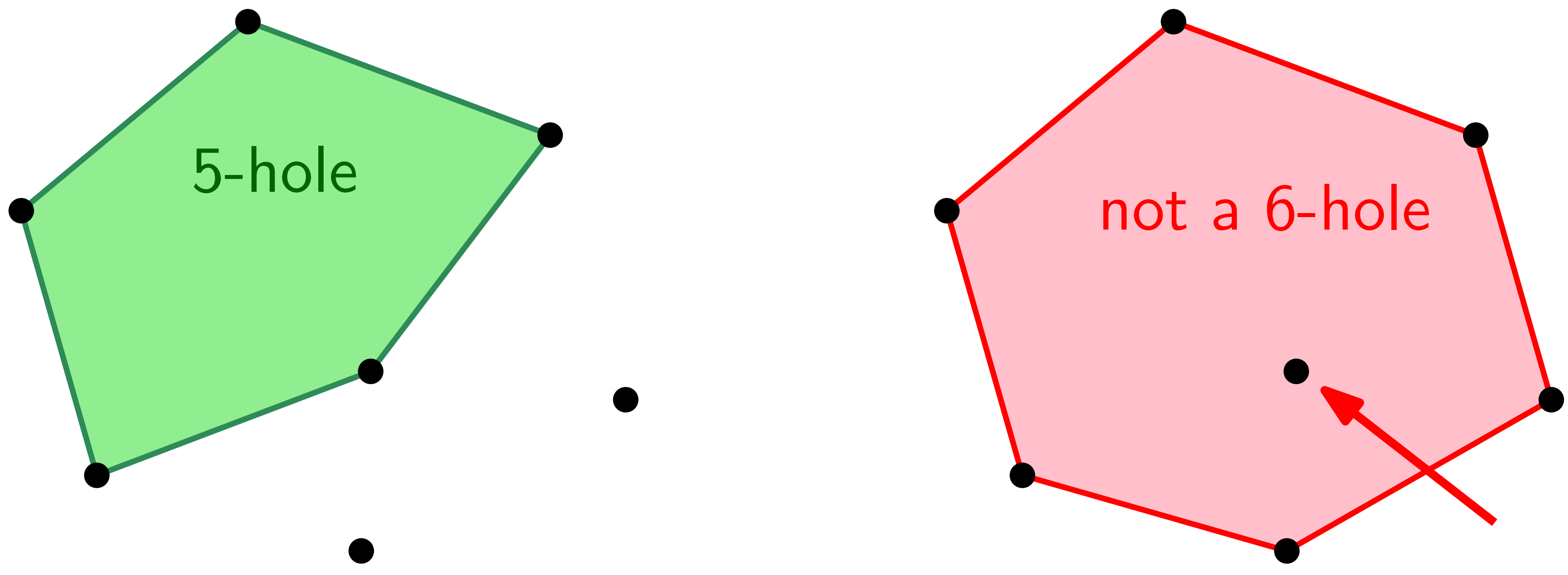
a k-hole (in S) is a k-gon which contains no other points of S



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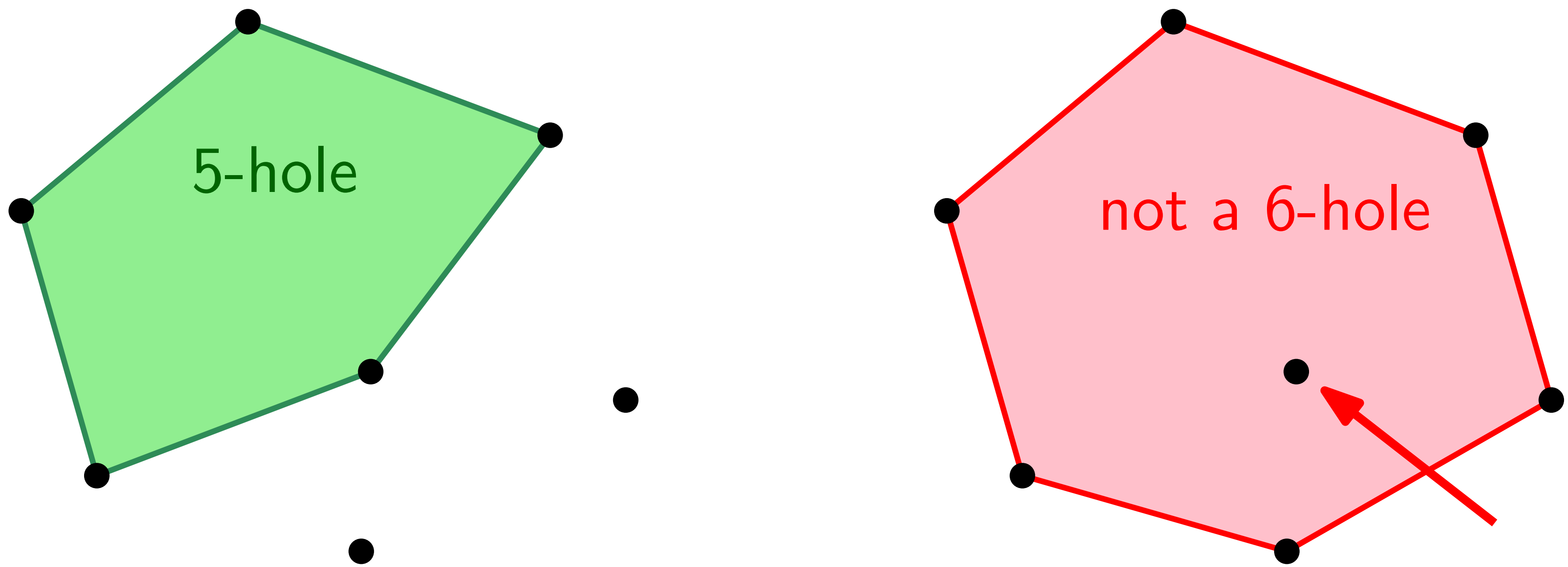
Erdős, 1970's: For k fixed, does every sufficiently large point set determine a k-hole?



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- 5 points $\Rightarrow \exists$ 4-hole
- 10 points $\Rightarrow \exists$ 5-hole [Harborth '78]

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  [Gerken '08 and Nicolás '07, independently]
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about 30 pages $\quad$ < 1 cpu hour using SAT (S. '22+)

k-Holes

- $h(4) = 5$, $h(5) = 10$, $30 \leq h(6) \leq g(9)$, $h(7) = \infty$

Harborth '78

Overmars '02

Gerken '08

Horton'83

k-Holes

exact value remains unknown

- $h(4) = 5$, $h(5) = 10$, $30 \leq h(6) \leq g(9)$, $h(7) = \infty$

Harborth '78

Horton'83

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Quantity of k-Gons

maximum # of k-gons among all sets of n points?

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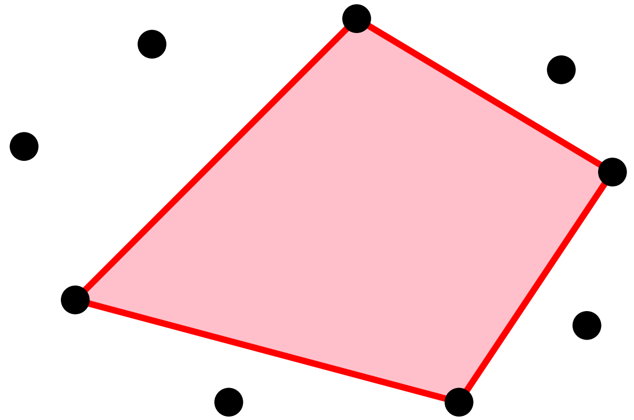

n points in convex position

any k-subset is k-gon

$$\Rightarrow \max = \binom{n}{k}$$

Quantity of k-Gons

$g_k(n)$:= minimum # of k-gons among all sets of n points

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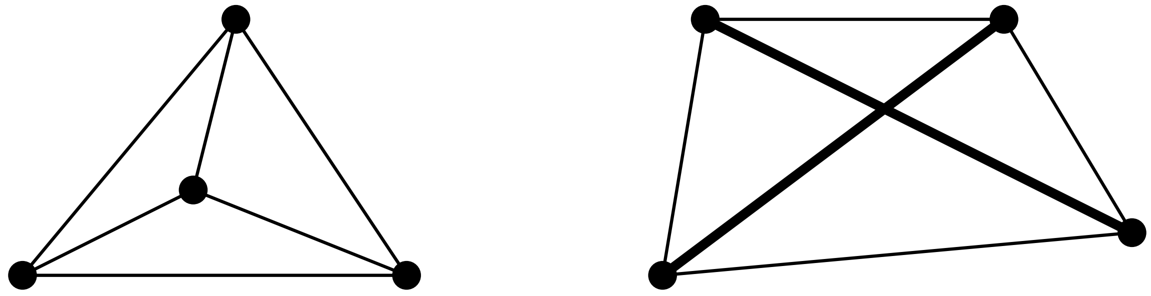
- $g_k(n) = \Theta(n^k)$
- $k = 4$: rectilinear crossing number of K_n:
  $g_4(n) = \overline{cr}(K_n) \sim c_4 \cdot \binom{n}{4}$ with $0.37999 < c_4 < 0.38055$
  [Ábrego et al. '08, Aichholzer et al. '20]



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- various notions of crossing numbers have been studied intensively (not necessarily straight-line drawings, not necessarily complete graphs)

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- h_3, h_4 both in $\Theta(n^2)$
- h_5 in $\Omega(n \log^{4/5} n)$ and $O(n^2)$

  [Aichholzer, Balko, Hackl, Kynčl, Parada, S., Valtr, and Vogtenhuber '17]
  (computer assisted proof, 20 pages)

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- $h_k(n) = 0$ for $k \geq 7$ [Horton '83]

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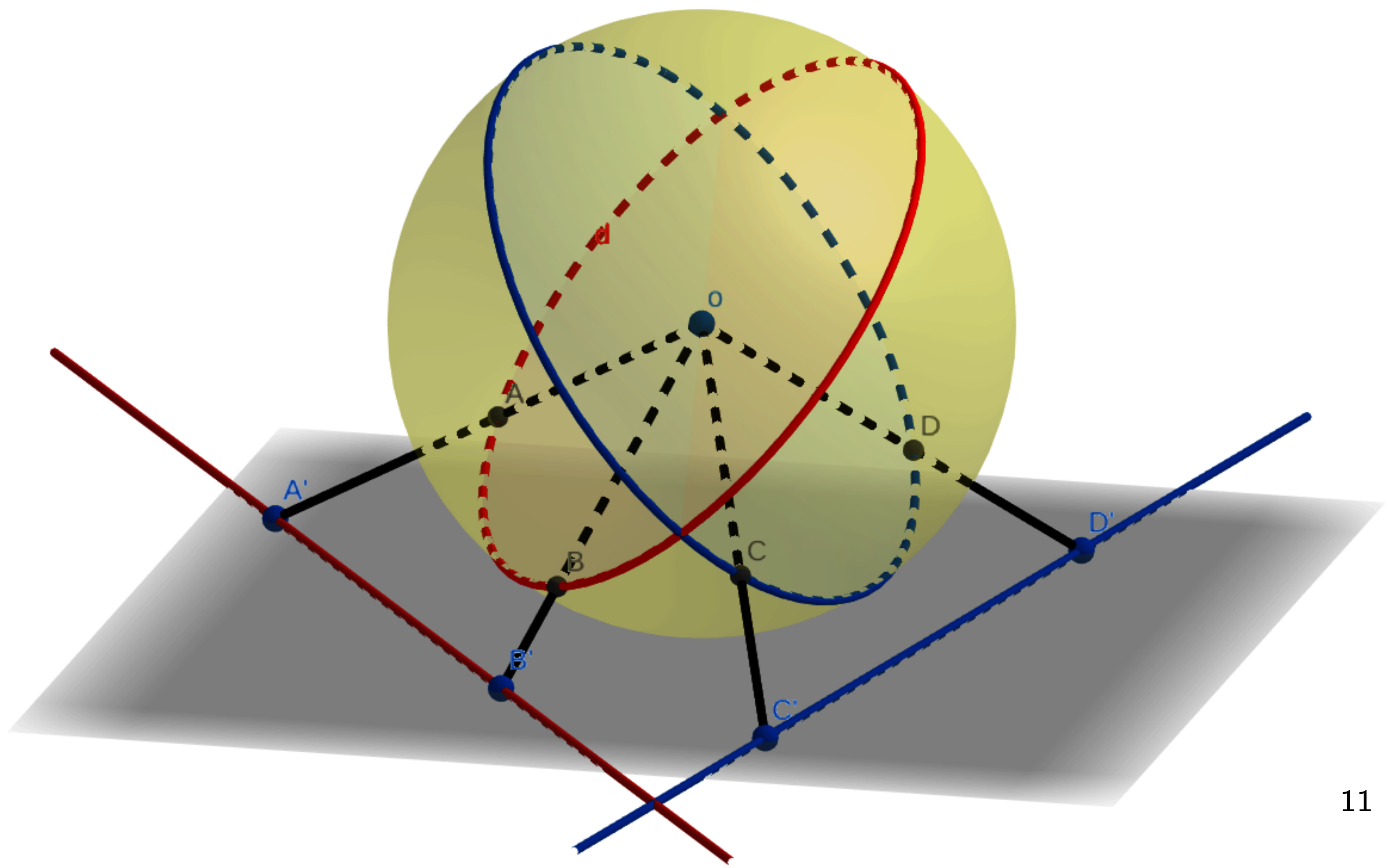
Conjecture:
h_5, h_6 are both in $\Theta(n^2)$

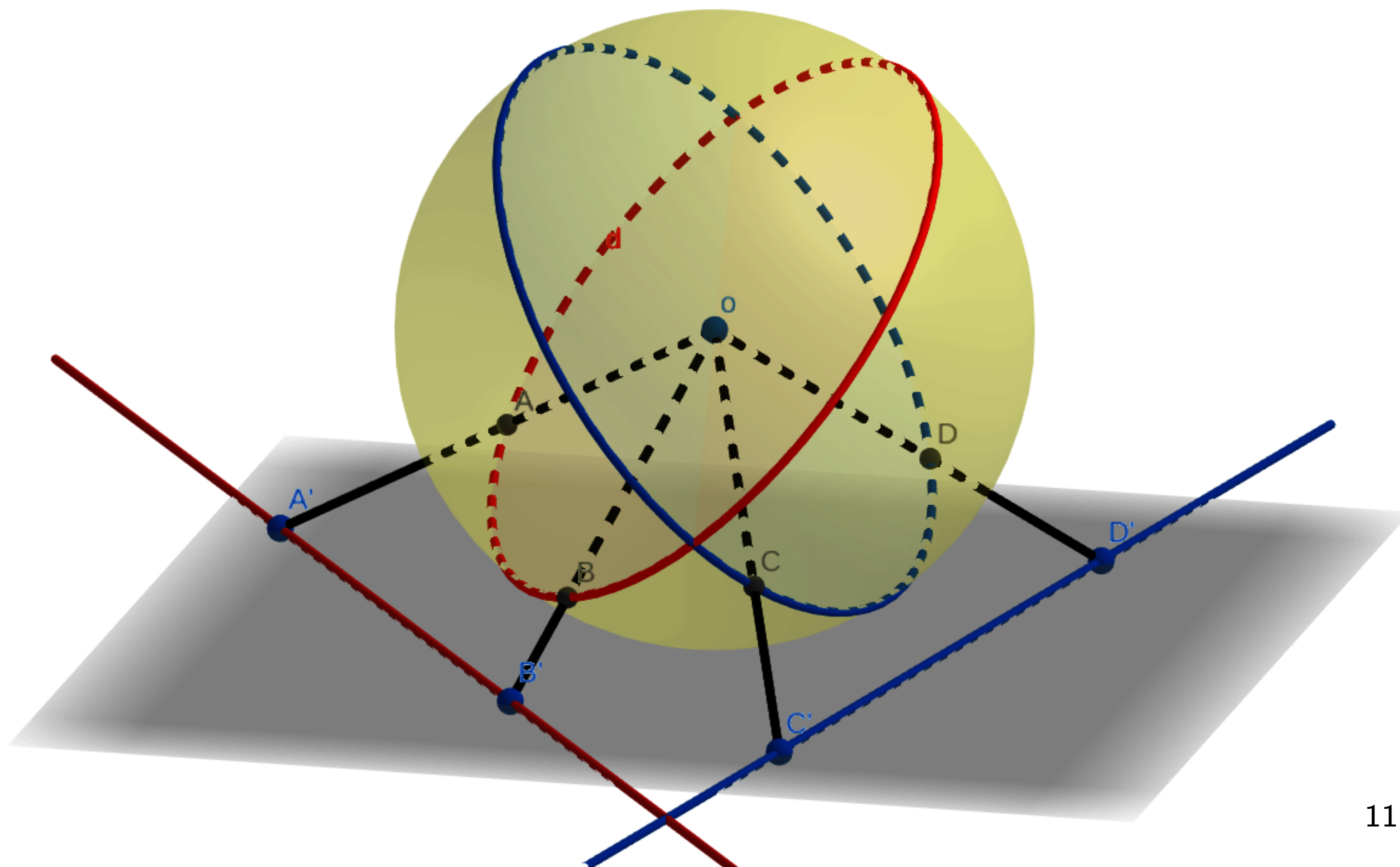
The Projective World

Convex Sets in Projective Plane (Steinitz 1913)



Convex Sets in Projective Plane (Steinitz 1913)

every pair of points p, q spans two *projective segments*:
an affine line segment pq and (its complement) $\overline{pq} \setminus pq$



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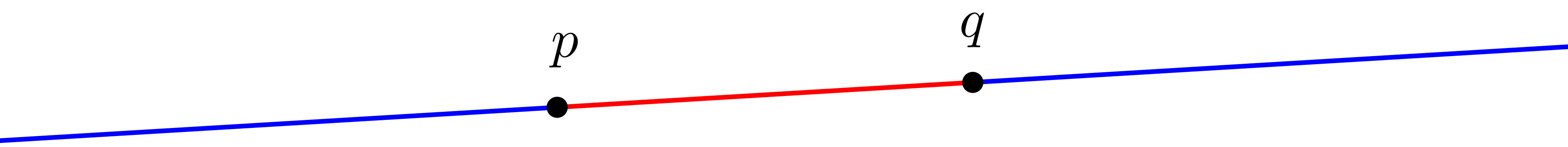

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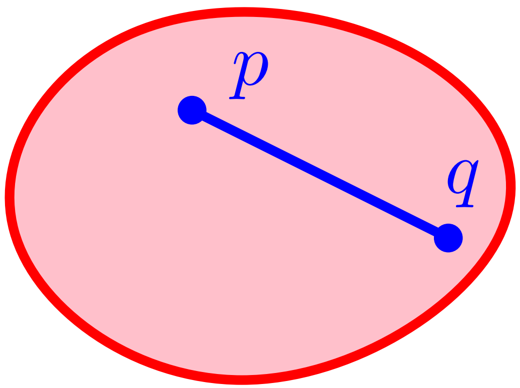
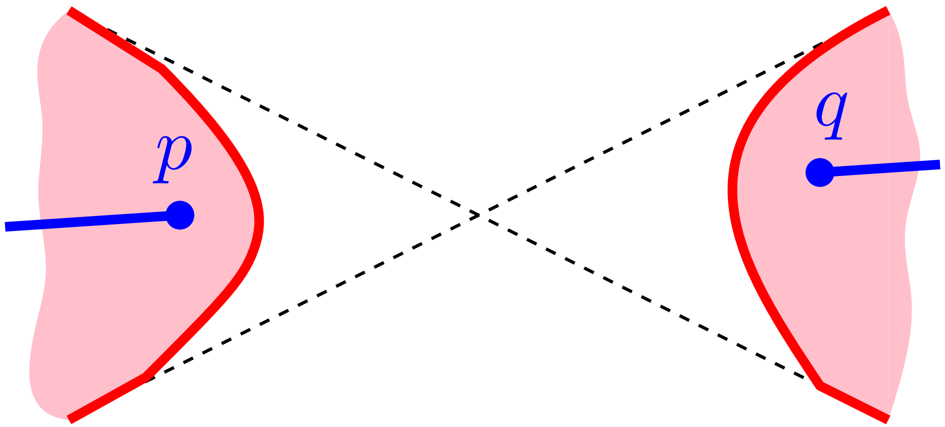
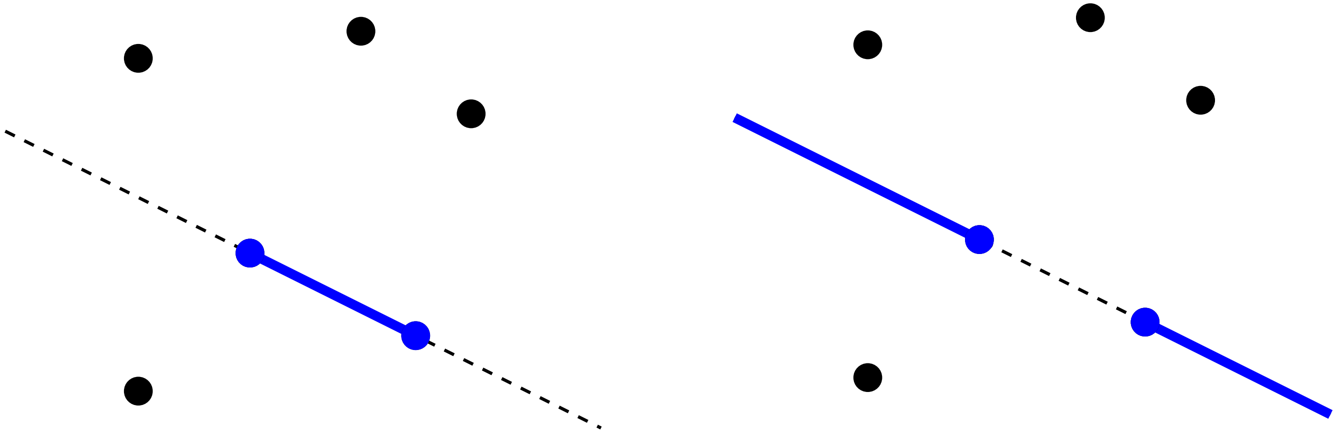
$C \subseteq \mathbb{RP}^2$ is projectively convex if, for every pair $p, q \in C$,
one of its projective segments is fully contained in C





Projective Convex Hull, Gons and Holes

projective convex hull: a inclusion-wise minimal convex set containing a given set P (not unique)



$|P| = 2 \Rightarrow 2$ p.c.h.s

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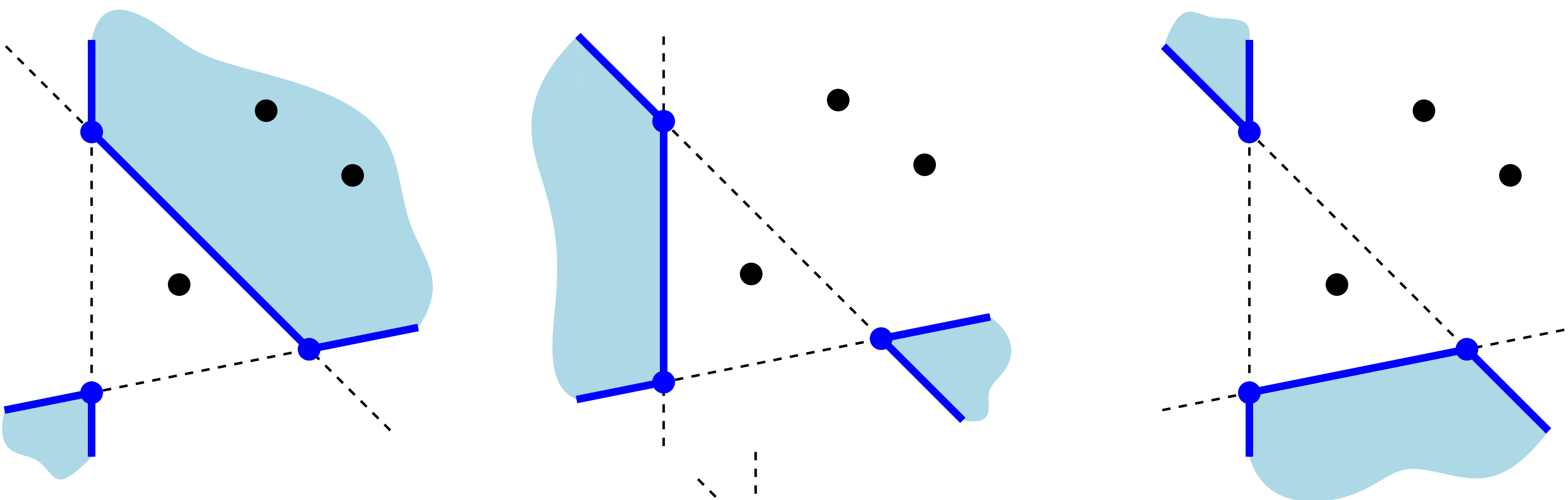
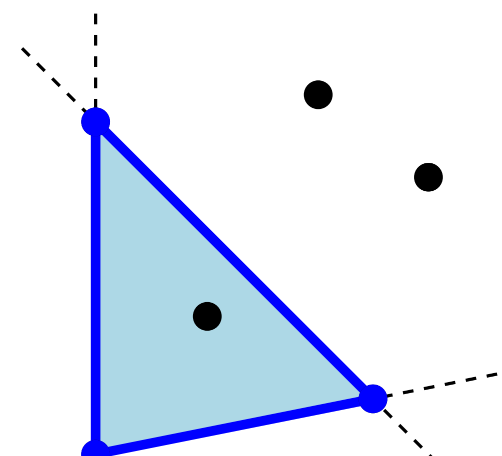


$|P| = 2 \Rightarrow 2$ p.c.h.s

$|P| = 3 \Rightarrow 4$ p.c.h.s



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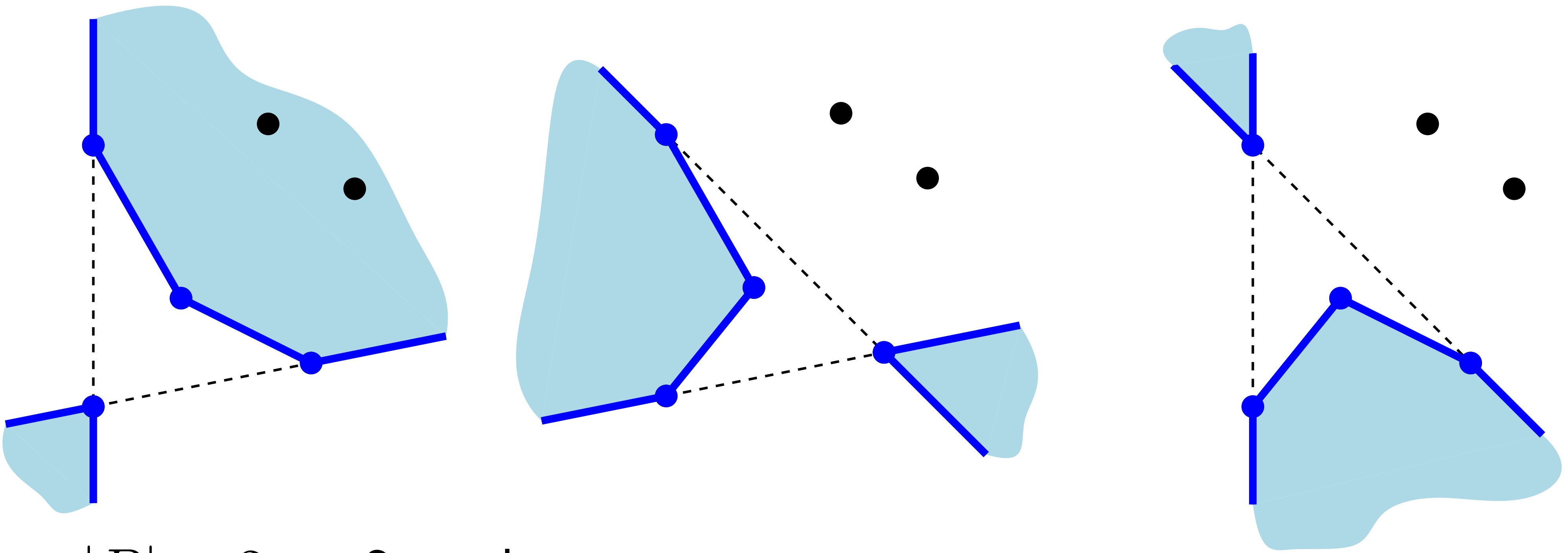

$|P| = 2 \Rightarrow$ 2 p.c.h.s

$|P| = 3 \Rightarrow$ 4 p.c.h.s

$|P| = 4 \Rightarrow$ 3 p.c.h.s (with all points on boundary)

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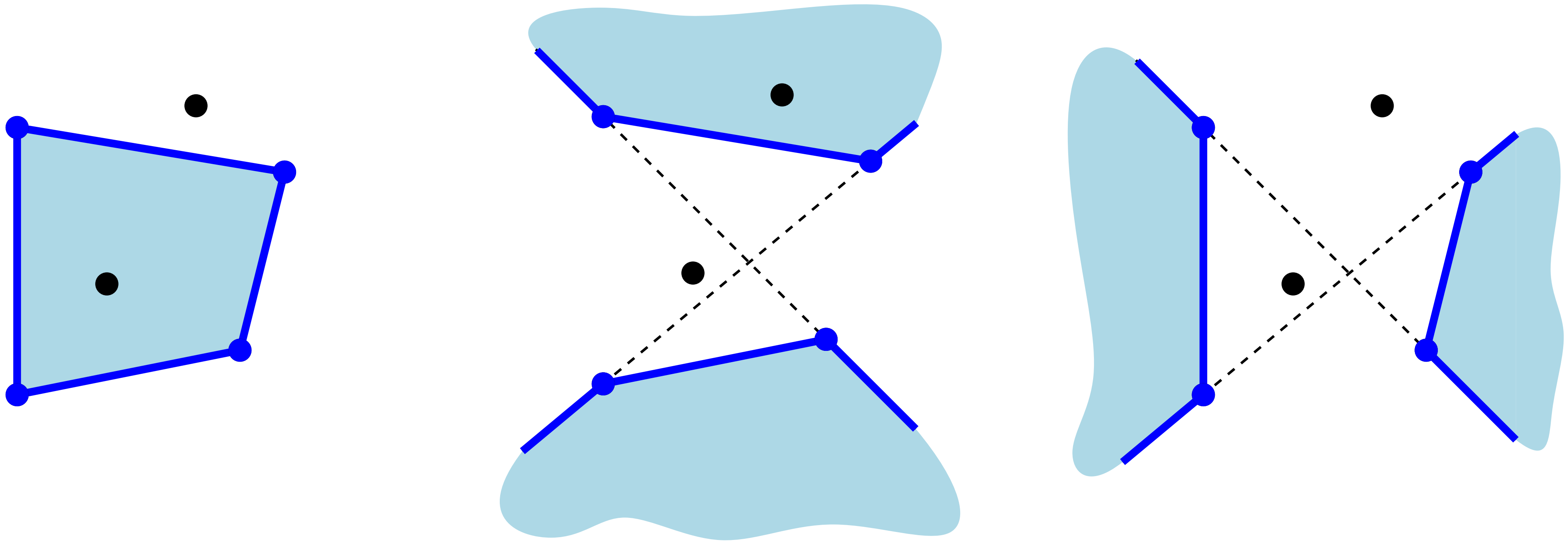

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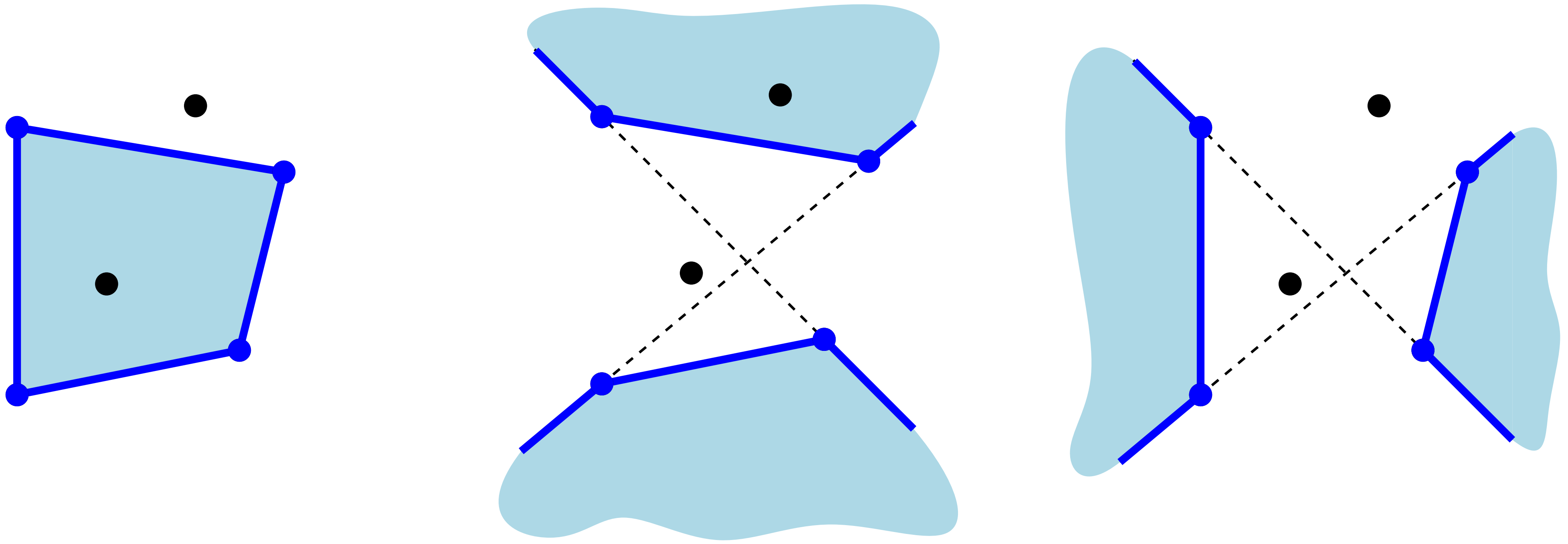

projective k-gon: projective convex set spanned by k pnts

(first introduced by Harborth & Möller '93)

Projective Convex Hull, Gons and Holes

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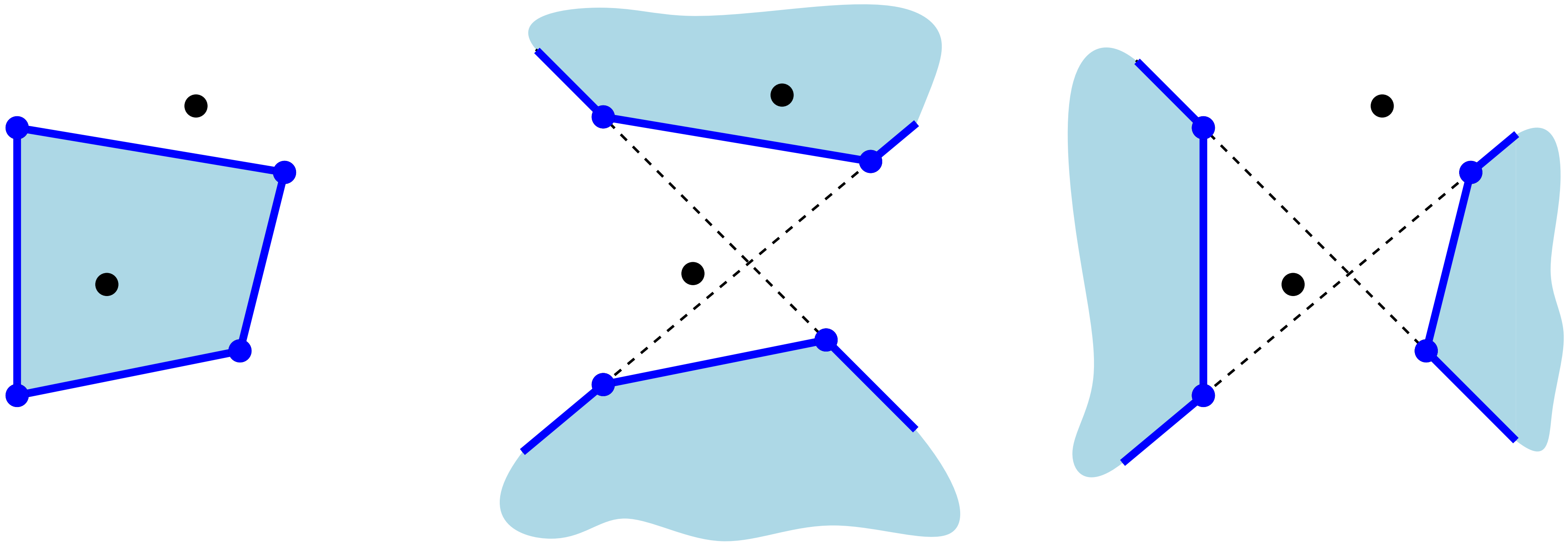

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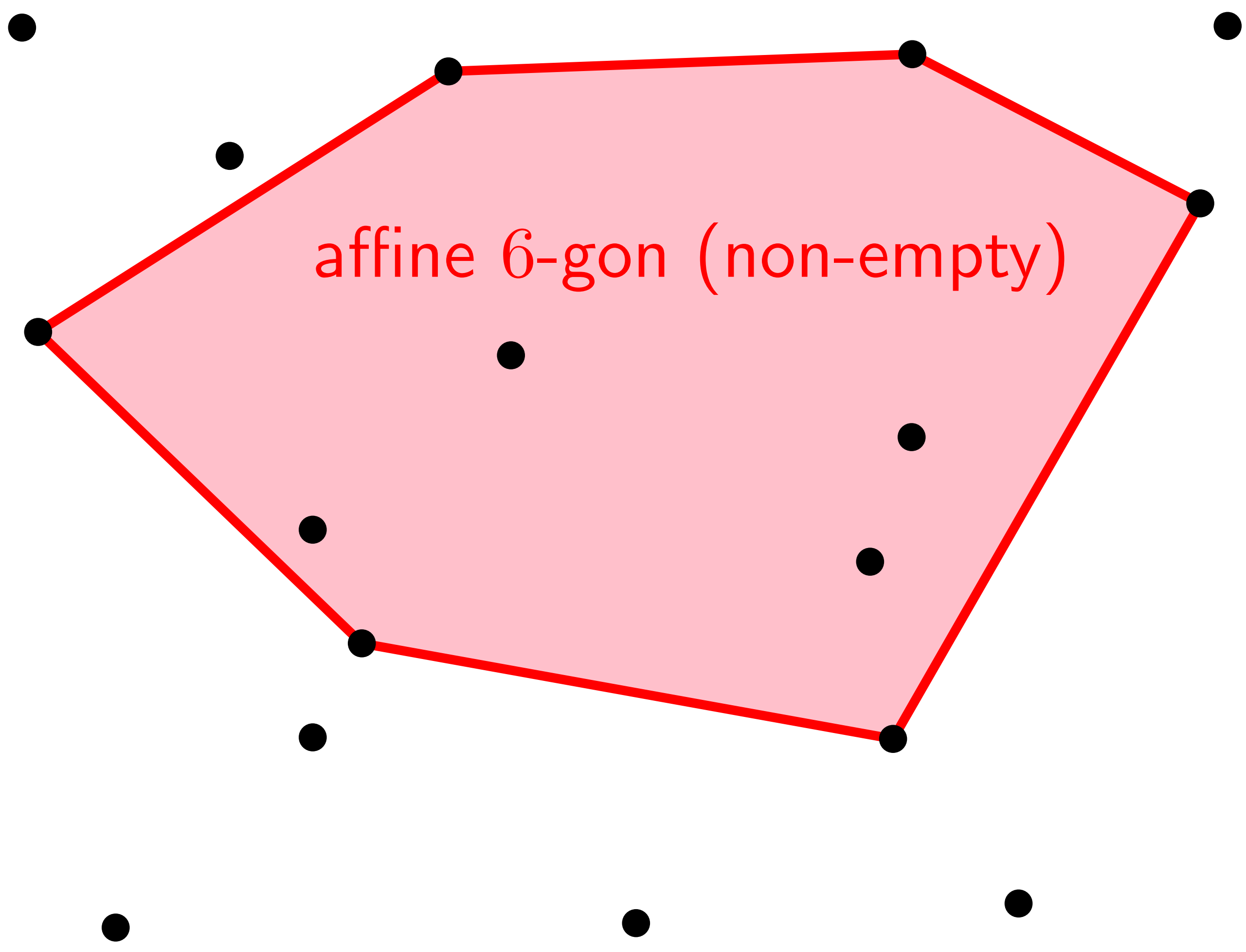
Projective Gons and Holes

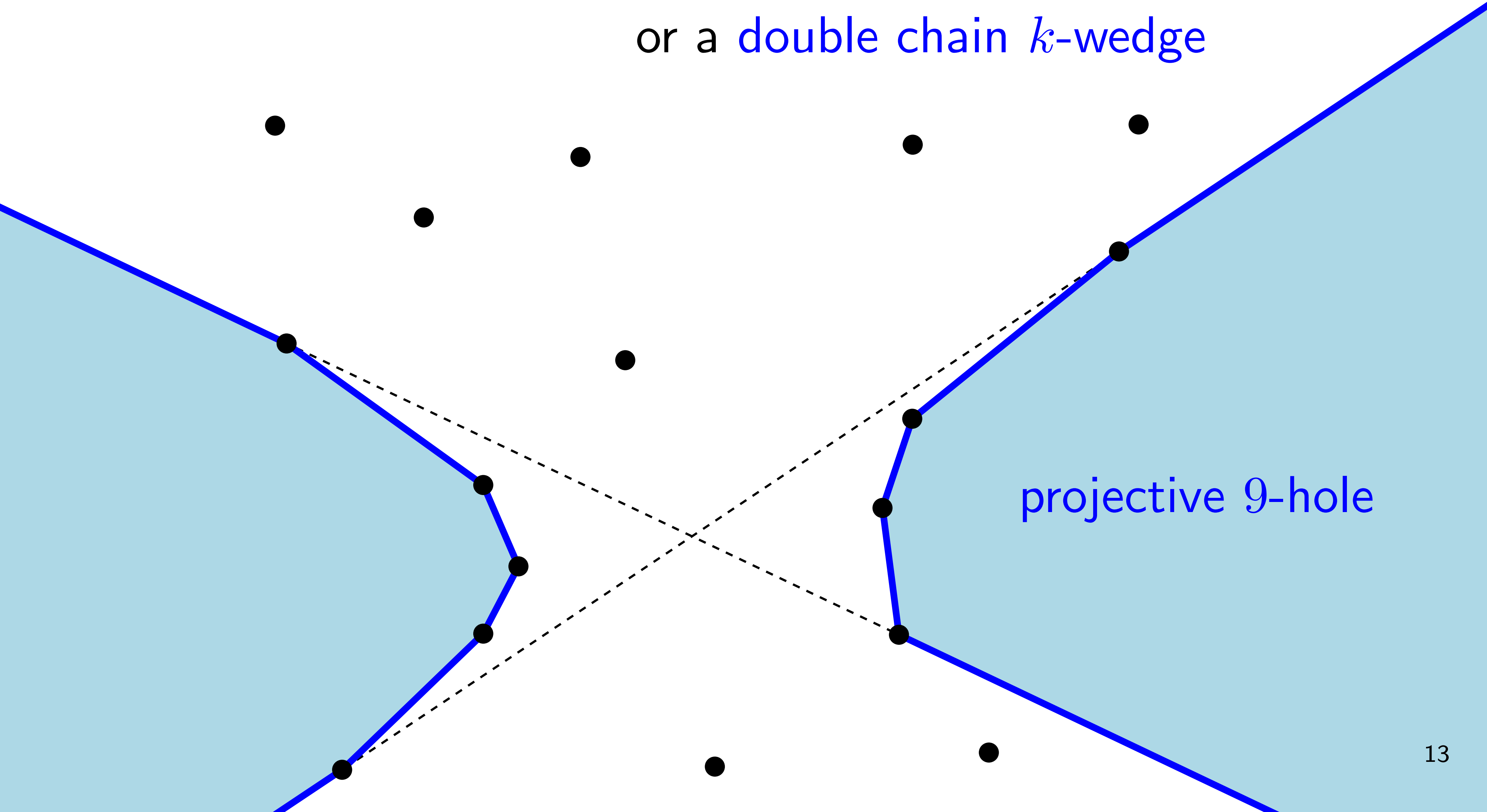
a projective k-gon is either an affine k-gon



Projective Gons and Holes

a projective k-gon is either an affine k-gon

or a double chain k-wedge



Erdős–Szekeres Numbers

Thm (affine k-gons; Erdős & Szekeres '35; Suk '16; Holmsen, Mojarrad, Pach & Tardos '17):

$$2^{k-2} + 1 \leq g(k) \leq 2^{k+O(\sqrt{k \log k})}$$

Thm (projective k-gons, BSV '22):

$$2^{k-O(\log k)} \leq g^p(k) \leq 2^{k+O(\sqrt{k \log k})}$$

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k-Holes in Horton Sets

Thm (affine holes, Horton '83, Bárány & Füredi '87):
Let S be a Horton set of size $n = 2^t$. Then
$h_k(S) \leq O(n^2)$ for $k \leq 6$ and $h_7(S) = 0$.

(h_k ... # of affine k-holes)

Thm (projective holes, BSV '22):
Let S be a Horton set of size $n = 2^t$. Then
$h_k^p(S) \leq O(n^2)$ for $k \leq 7$ and $h_8^p(S) = 0$.

(h_k^p ... # of projective k-holes)

k-Holes Summarized

$k = 3$: always exist

$k = 4$: $h_4(\geq 5) \geq 1 \longrightarrow h_4^p(\geq 4) \geq 1$

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Harborth '78

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Harborth '78

weird behavior

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$k = 6$:
Gerken '08
$h_6(\geq g(9)) \geq 1$
$\longrightarrow h_6^p(\geq g^p(9)) \geq 1$

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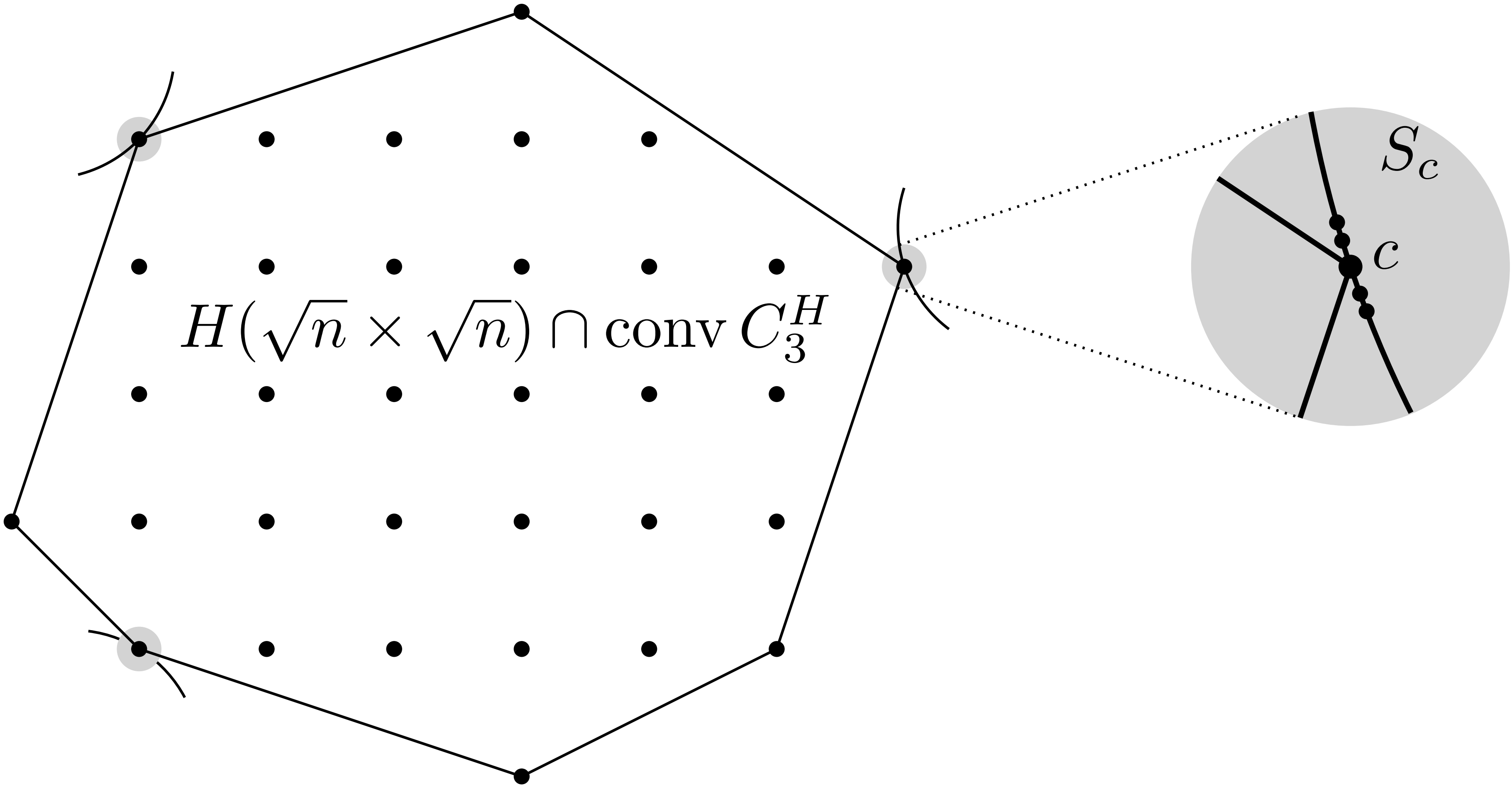
$k = 6$:
$h_6(\geq g(9)) \geq 1$
$\longrightarrow h_6^p(\geq g^p(9)) \geq 1$

$k \geq 8$: not exist (Horton sets)

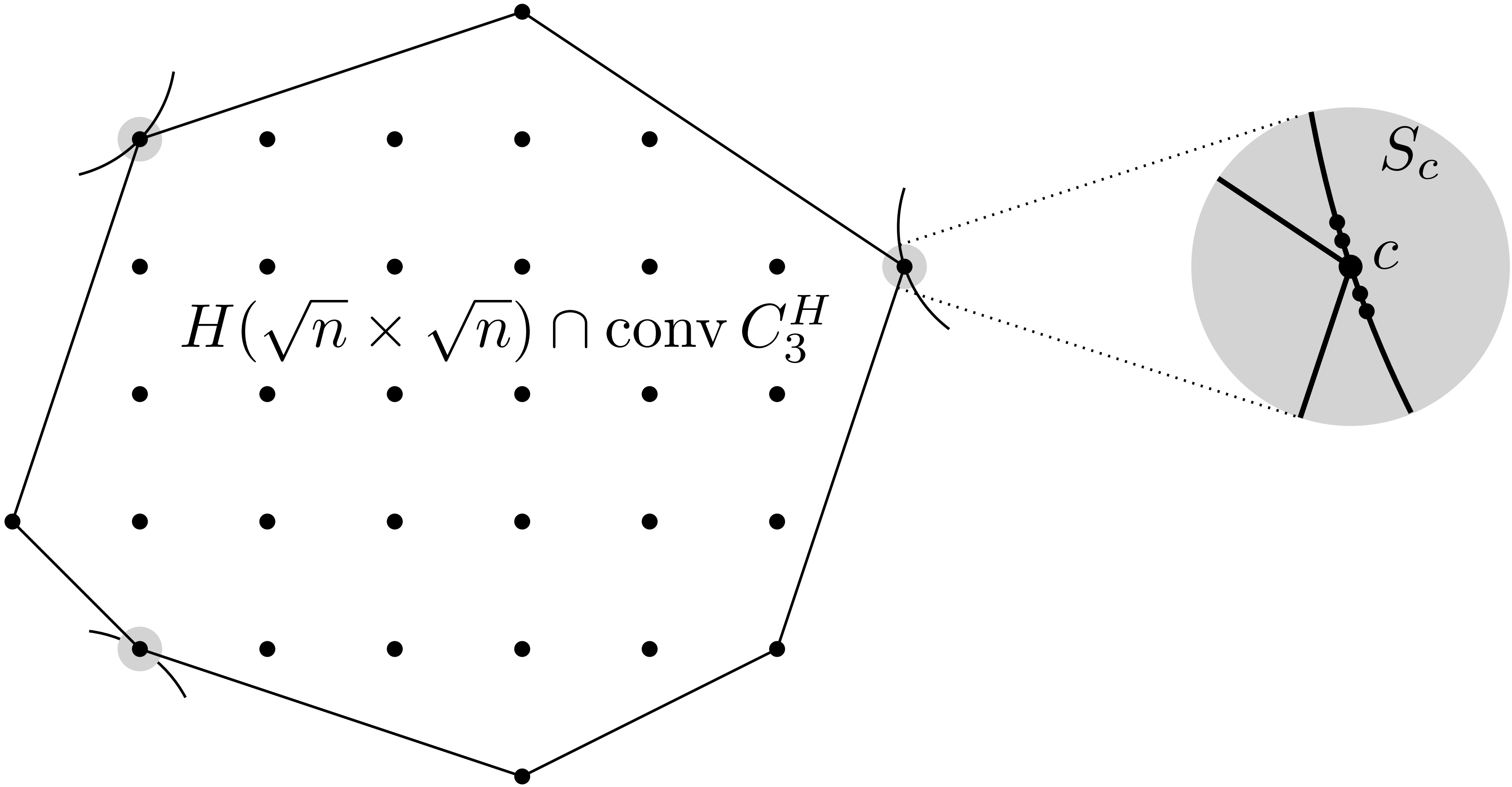
$k = 7$: no affine, but projective 7-holes in Horton sets.
existence of projective 7-holes remains open!

Any significant difference?

Substantially more Projective Holes

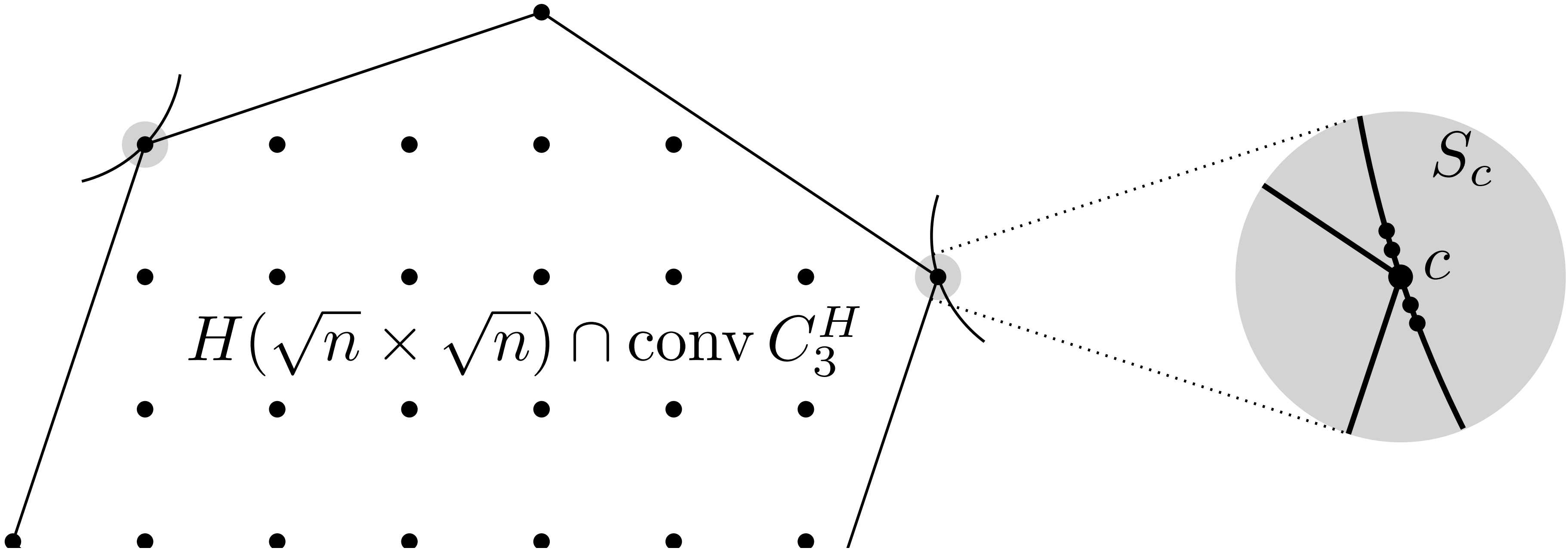


Substantially more Projective Holes



Thm (BSV '22). $\forall\, k \in \{3, \ldots, 6\}$ and n, $\exists$ n-point set with $O(n^2)$ affine k-holes and $\Omega(n^{3-\frac{5}{3k}})$ projective k-holes.

Substantially more Projective Holes



Thm (BSV '22) $\forall$ n and $x \leq 2^{n/2}$ $\exists$ n-point set with $O(x + n^2)$ affine holes and $\Omega(x^2)$ projective holes.

Thm (BSV '22). $\forall$ $k \in \{3, \dots, 6\}$ and n, $\exists$ n-point set with $O(n^2)$ affine k-holes and $\Omega(n^{3-\frac{5}{3k}})$ projective k-holes.

Further Results

Holes in Random Point Sets

Affine:

$EH_3 \sim 2n^2$ [Valtr '95, Reitzner & Temesvari '19]

$EH_k = \Theta(n^2)$ [BSV '19+'21]

Projective [BSV '22]:

$EH_3^p = \Theta(n^2)$ with larger multiplicative constant

Holes in Random Point Sets

Affine:

$EH_3 \sim 2n^2$ [Valtr '95, Reitzner & Temesvari '19]

$EH_k = \Theta(n^2)$ [BSV '19+'21]

Projective [BSV '22]:

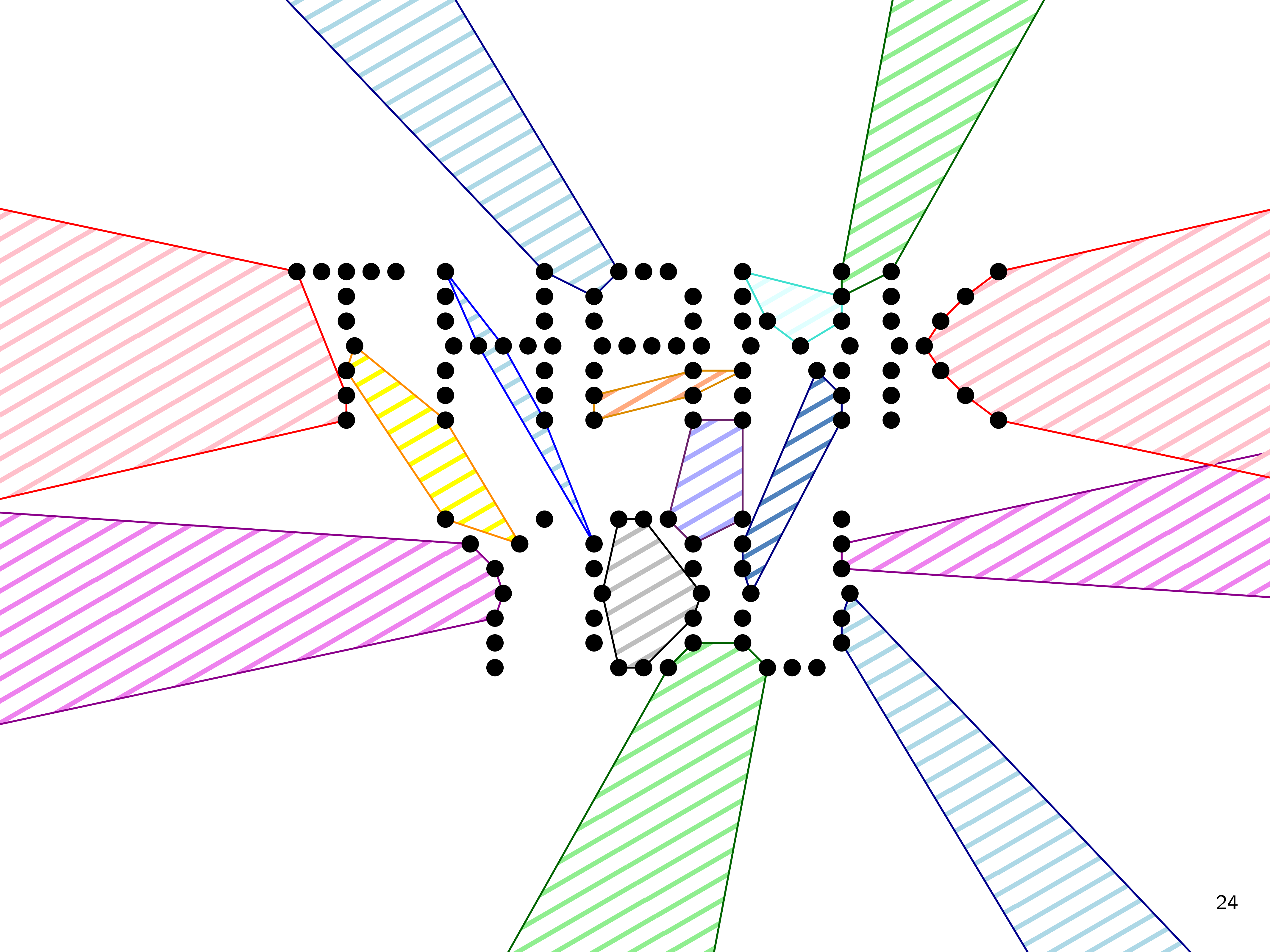
$EH_3^p = \Theta(n^2)$ with larger multiplicative constant

no proof for larger holes, but $\Theta(n^2)$ conjectured!

Algorithmic Aspects

Thm (Mitchell, Rote, Sundaram & Woeginger '95).
The number of affine k-gons and k-holes in an n-point set can be computed in $O(kn^3)$ time and $O(kn^2)$ space.

Thm (BSV '22).
The number of projective k-gons and k-holes in an n-point set can be computed in $O(kn^4)$ time and $O(kn^2)$ space.


THANK
YOU