Orthogeodesic Point Set Embeddings of Outerplanar Graphs

Manfred Scheucher
Graz, June 18, 2015
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Motivation

Figure: Point Set Embedding of a Tree
Motivation

Figure: Point Set Embedding of a Tree

Table: Upper bounds given by Giacomo et al.

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Graph

- A graph is a tuple $G = (V, E)$ with

$$ E \subseteq \{\{u, v\} | u \neq v \in V\}.$$

- $V$ is said to be the set of vertices and $E$ the set of edges.
- An example:

$$K_3 = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\})$$
Embedding of a Graph

- An **embedding** of a graph $G$ is a tuple $(\nu, \mu)$ where
  - $\nu$ is an injective mapping of the vertices into the plane
  - and $\mu$ maps every edge $e = \{u, v\}$ to a polygonal arc with endpoints $\nu(u)$ and $\nu(v)$

![Figure: Two embeddings of $K_3$.](image-url)
Embedding of a Graph

- An embedding is said to be **planar** if the interior of every edge neither intersects other edges nor contains vertices.

Figure: A nonplanar, a planar, and a straight-line planar embedding of $K_4$. 
Point Set Embedding

- Let $P \subset \mathbb{R}^2$ be a set of points. We call an embedding with $\nu(V) \subseteq P$ an embedding in $P$ (PSE).

- Planar PSE in every point set of size $n$ with at most two bends per edge [Kaufmann and Wiese, 1999]

- Deciding whether a planar PSE with at most one bend per edge exists is NP-complete [Kaufmann and Wiese, 1999]
A PSE is said to be **orthogeodesic** if

- every edge (polygonal arc) has minimal $L^1$-length
- edges are drawn on the grid of horizontal and vertical lines induced by the points in $P$
- all edges incident to a vertex enter from distinct directions

Figure: Orthogeodesic embedding of $K_3$. 
Orthogeodesic PSE

Figure: Edges in planar orthogeodesic embeddings.
An orthogeodesic PSE is said to be **L-shaped** if every edge has at most one bend.
Deciding whether an orthogeodesic PSE exists is NP-complete. [Katz et al., 2010]
Orthogeodesic PSE

- Restriction to general point sets
- Restriction to certain classes of graphs
General Point Set

- A point set $P \subseteq \mathbb{R}^2$ is said to be **general** if each two points have distinct $x$- and $y$-coordinates.
General Point Set

- A point set $P \subset \mathbb{R}^2$ is said to be **general** if each two points have distinct $x$- and $y$-coordinates.

- W.l.o.g.,

  \[ x_1 < x_2 < \ldots < x_n \]

  and

  \[ y_{\sigma_1} < y_{\sigma_2} < \ldots < y_{\sigma_n} \]

  hold for a permutation $\sigma$. 

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General Point Set

- A point set $P \subset \mathbb{R}^2$ is said to be **general** if each two points have distinct $x$- and $y$-coordinates.
- W.l.o.g.,
  $$x_1 < x_2 < \ldots < x_n$$
  and
  $$y_{\sigma_1} < y_{\sigma_2} < \ldots < y_{\sigma_n}$$
  hold for a permutation $\sigma$.
- W.l.o.g.,
  $$P = \{(1, \pi_1), (2, \pi_2), \ldots, (n, \pi_n)\}$$
  holds for a permutation $\pi$. (Actually, $\pi = \sigma^{-1}$)
General Point Set

Figure: General point sets up to size 4 (+symmetry).
Diagonal Point Sets

- Every point set of size $n^2 + 1$ admits a diagonal point set of size $n + 1$ [Erdős and Szekeres, 1935]

Figure: An illustration.
If $G$ admits an embedding in a diagonal point set of size $n$, then it admits an embedding in every point set of size $(n - 1)^2 + 1$.

Otherwise, it can not be embedded in certain point sets (e.g., in diagonal point sets).
Classes of Graphs

- Planar graphs
- Outerplanar graphs
- Trees
- Caterpillars

Figure: An outerplanar graph, a tree, and a caterpillar.
Classes of Graphs

- $\Delta(G) \leq 4$ necessary, but not sufficient
Classes of Graphs

- $\Delta(G) \leq 4$ necessary, but not sufficient
- Planar with $\Delta \in \{3, 4\}$: NO
- Outerplanar with $\Delta = 4$: NO
- Outerplanar with $\Delta = 3$:
  - Planar L-shaped: NO
Classes of Graphs

- $\Delta(G) \leq 4$ necessary, but not sufficient
- Planar with $\Delta \in \{3, 4\}$: NO
- Outerplanar with $\Delta = 4$: NO
- Outerplanar with $\Delta = 3$:
  - Planar L-shaped: NO
  - L-shaped: YES
  - Planar orthogeodesic: YES
- Trees and caterpillars: YES [Giacomo et al., 2013]
What about Point Sets of Subquadratic Size?
What about Point Sets of Subquadratic Size?

- For caterpillars: $O(n)$ [Giacomo et al.]
What about Point Sets of Subquadratic Size?

- For caterpillars: $O(n)$ [Giacomo et al.]
- For trees: Recursive Embedding

![Diagram showing recursion layers 1, 2, and 3]
Embedding 3-Trees

- Start with root (e.g., a leaf) and continue recursively

Figure: Recursive embedding of a 3-tree.
Recursive Embedding Techniques

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Embedding 3-Trees

- Start with root (e.g., a leaf) and continue recursively

\[ f(n) \geq 1 + f(a) + 2f(b) \text{ for } a \geq b \text{ and } a + b = n - 1 \]

Figure: Recursive embedding of a 3-tree.
Recursive Embedding Techniques

Embedding 3-Trees

- Start with root (e.g., a leaf) and continue recursively

\[
f(n) \geq 1 + f(a) + 2f(b) \text{ for } a \geq b \text{ and } a + b = n - 1
\]

- Trivial solution: \( f(n) = n^2 \)

Figure: Recursive embedding of a 3-tree.
Consider $f$ convex with $f(0) = 0$ and

$$f(n) \geq \max_{0 \leq b \leq \frac{n-1}{2}} 1 + f(n - 1 - b) + 2f(b)$$

$$=: \phi_n(b)$$
Embedding 3-Trees

- Consider $f$ convex with $f(0) = 0$ and
  \[
  f(n) \geq \max_{0 \leq b \leq \frac{n-1}{2}} 1 + f(n - 1 - b) + 2f(b)
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  \[= : \phi_n(b) \]

- sum of convex functions is convex
Recursive Embedding Techniques

Embedding 3-Trees

- Consider \( f \) convex with \( f(0) = 0 \) and

\[
f(n) \geq \max_{0 \leq b \leq \frac{n-1}{2}} \left( 1 + f(n - 1 - b) + 2f(b) \right)
\]

= \( \phi_n(b) \)

- sum of convex functions is convex
- convex function on convex set (Maximum Principle)

Figure: An illustration of the \( \phi_n \) function.
Embedding 3-Trees

- \( f(n) \geq \max \{ \phi_n(0), \phi_n\left(\frac{n-1}{2}\right) \} = \phi_n\left(\frac{n-1}{2}\right) = 3f\left(\frac{n-1}{2}\right) + 1 \), since

\[
\phi_n(0) = f(n-1) + 1 \leq f(n) \leq f'(\xi)
\]
Recursive Embedding Techniques

Embedding 3-Trees

- \( f(n) \geq \max\{\phi_n(0), \phi_n\left(\frac{n-1}{2}\right)\} = \phi_n\left(\frac{n-1}{2}\right) = 3f\left(\frac{n-1}{2}\right) + 1 \), since
  \[
  \phi_n(0) = f(n - 1) + \underbrace{1}_{\leq f'(\xi)} \leq f(n)
  \]

- A solution: \( f(n) = n^{\log_2 3} \), where \( \log_2 3 = 1.5849 \ldots \)
  because \( 3f\left(\frac{n-1}{2}\right) + 1 = \frac{3}{3}(n - 1)^{\log_2 3} = f(n - 1) + 1 \)
Recursive Embedding Techniques

Embedding 3-Trees

- $f(n) \geq \max\{\phi_n(0), \phi_n(\frac{n-1}{2})\} = \phi_n(\frac{n-1}{2}) = 3f(\frac{n-1}{2}) + 1$,
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- A solution: $f(n) = n^{\log_2 3}$, where $\log_2 3 = 1.5849 \ldots$
  because $3f(\frac{n-1}{2}) + 1 = \frac{3}{3}(n - 1)^{\log_2 3} = f(n - 1) + 1$

Theorem

$f(n) = O(n^{\log_2 3})$
Embedding 4-Trees

- Analogous

\[
f(n) \geq 1 + f(a) + 2f(b) + 2f(c) \text{ with } a \geq b \geq c \ldots
\]

\[
f(n) = O(n^{\log_2 3})
\]
Recursive Embedding Techniques

Embedding 4-Trees

- Consider $f$ convex with $f(0) = 0$ and

\[
f(n) \geq \max_{0 \leq c \leq b} \max_{b \leq n-1-b-c} 1 + f(n-1-b-c) + 2f(b) + 2f(c) =: \phi_n(b,c)
\]

- Maximum Principle: analyze the corners of the convex set

\[
C = \{(b, c) \mid 0 \leq c \leq b \leq n-1-b-c\}
\]
Recursive Embedding Techniques

Embedding 4-Trees

Figure: Maximum Principle illustration.
Embedding 4-Trees

Figure: Corners of the convex set $C$. 

Recursive Embedding Techniques

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## Results for the General Case

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**Table : Best upper bounds currently known**
What else can we do?

- Analyze Trees
- Analyze Point Sets
- Probabilistic Analysis
Saturation Property

\[ \sigma_{T,r}(v) := \max\{0, \sigma_{T,r}(u_1), \sigma_{T,r}(u_2) + 1, \ldots, \sigma_{T,r}(u_k) + 1\} \]
Saturation Property

\[ \sigma_{T,r}(v) := \max\{0, \sigma_{T,r}(u_1), \sigma_{T,r}(u_2) + 1, \ldots, \sigma_{T,r}(u_k) + 1\} \]
Further Results

Saturation Property

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Saturation Property

\[ \sigma_{T,r}(v) \coloneqq \max\{0, \sigma_{T,r}(u_1), \sigma_{T,r}(u_2) + 1, \ldots, \sigma_{T,r}(u_k) + 1\} \]
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\[ \sigma_{T,r}(v) := \max\{0, \sigma_{T,r}(u_1), \sigma_{T,r}(u_2) + 1, \ldots, \sigma_{T,r}(u_k) + 1\} \]
Further Results

Saturation Property

- $\sigma(T) := \min_{r \in V} \sigma_{T,r}(r)$
- $f(T) = O(n \cdot 2^{\sigma(T)})$
- For caterpillars:
  \[
  \sigma(T) = O(1) \Rightarrow f(T) = O(n)
  \]
- For trees:
  \[
  \sigma(T) \leq \log_2(n + 1) \Rightarrow f(T) = O(n^2)
  \]
Orthogonal Convex Hull
Further Results

Orthogonal Convex Hull

- $l$ . . . number of layers in onion peeling
- $k_i$ . . . number of points in layer $i$
- $P$ contains diag. PS of size

$$n := \max \left\{ 2l - 1, \left\lfloor \frac{k_1}{4} \right\rfloor, \ldots, \left\lfloor \frac{k_l}{4} \right\rfloor \right\}$$
Orthogonal Convex Hull

- $l \ldots$ number of layers in onion peeling
- $k_i \ldots$ number of points in layer $i$
- $P$ contains diag. PS of size

\[ n := \max \left\{ 2l - 1, \left\lfloor \frac{k_1}{4} \right\rfloor, \ldots, \left\lfloor \frac{k_l}{4} \right\rfloor \right\} \]

- Yet another proof of the $m = O(n^2)$ bound, because $n = \Omega(\sqrt{m})$
Further Results

Orthogonal Convex Hull

(a) $l$ layers in onion peeling

(b) Layer $i$ of size $k_i$
Probabilistic Results

- Point exists with probability at least $1 - \left(1 - \frac{|A|}{|P|}\right)^{|C|}$
Point exists with probability at least \( 1 - \left( 1 - \frac{|A|}{|P|} \right)^{|C|} \)

- For 3-Trees: \( O(n \log n (\log \log n)^2) \)
- For 4-Trees: \( O(n^{\gamma_0 + \varepsilon}) \) where \( \gamma_0 = 1.3319 \cdots \)
### Summary

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- \( f_{LT4}(T) = O(n \cdot 2^{\sigma(T)}) \)
- \( n \geq \max \left\{ 2l - 1, \left[ \frac{k_1}{4} \right], \ldots, \left[ \frac{k_l}{4} \right] \right\} \)
- \( f_{LT3}^{1/2}(n) = O(n \log n (\log \log n)^2) \)
- \( f_{LT4}^{1/2}(n) = O(n^{\gamma_0 + \varepsilon}) \) where \( \gamma_0 = 1.3319 \cdots \)
Thank you for your attention!
(a) An outerplanar graph with $\Delta = 4$ that does not admit an embedding

(b) Sketch of the proof.
Classes of Graphs

(a) An outerplanar graph with $\Delta = 3$ that does not admit a planar L-shaped embedding

(b) A planar graph with $\Delta = 3$ that does not admit an embedding
Figure: Outerplanar Graphs $\Delta = 3$ (L-Shaped)
Figure: Outerplanar Graphs $\Delta = 3$ (Planar Orthog.)
Recursive Embeddings

- \( f_{OT4}(n) \leq 2n \)
- Actually, \( f_{OT4}(n) \leq \frac{3}{2}n - 1 \), because

\[
\left| \{ v \in V : \deg(v) \geq 3 \} \right| \leq \frac{n-2}{2}
\]
Recursive Embeddings

- With $U := \{v \in V : \deg(v) \geq k\}$

$$2n - 2 = 2|E| = \sum_{v \in V} \deg(v) \geq n + (k - 1)|U|,$$

- or equally,

$$|U| \leq \frac{n - 2}{k - 1}$$
Proof of $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$

- Ring-partition
- Case 1: recycle points, $O(1)$ wasted points
- Case 2: At most 2 wasted point per vertex
- Case 3: At most 4 wasted points per vertex
Proof of $f_{NT4}(n) \leq \frac{7}{3} n + O(1)$

maximize $2x_3 + 4x_4$

subject to $\sum_{i=1}^{4} x_i = n$

$\sum_{i=1}^{4} (i - 2)x_i = -2$ // holds for every tree

$x_i \in \mathbb{N}_0$, $1 \leq i \leq 4$

- $x_2^* = 0$ must hold . . .
Proof of $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$

maximize $2x_3 + 4x_4$

subject to $x_1 + x_3 + x_4 = n$

$x_1 = 2 + x_3 + 2x_4$

$x_1, x_3, x_4 \geq 0$
Proof of $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$

maximize $2x_3 + 4x_4$

subject to $2x_3 + 3x_4 = n - 2$

$2 + x_3 + 2x_4 \geq 0$

$x_3, x_4 \geq 0$
Proof of $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$

maximize $n - 2 + x_4$
subject to $3x_4 \leq n - 2$

$x_4 \geq 0$

- number of wasted points at most $\frac{4}{3}n + O(1)$
- $f_{NT4}(n) \leq \frac{7}{3}n + O(1)$
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**Table: Best upper bounds currently known**
Point exists with probability at least

$$1 - \left(1 - \frac{|A|}{|P|}\right)^{|C|}$$
Probabilistic Results

- \( \mathbb{P}(\bigcup_{i=1}^{h} E_i) = 1 - \mathbb{P}(\bigcap_{i=1}^{h} \overline{E}_i) = 1 - \prod_{i=1}^{h} \mathbb{P}(\overline{E}_i) \cap \bigcap_{j=1}^{i-1} \overline{E}_j) \)

- \( \mathbb{P}(\overline{E}_1) = 1 - \frac{|A| + 1}{|A| + |B| + 1} \)

- \( \mathbb{P}(\overline{E}_i) \cap \bigcap_{j=1}^{i-1} \overline{E}_j) = \frac{|B| + (i - 1)}{|A| + |B| + (i - 1) + 1} = 1 - \frac{|A| + 1}{|A| + |B| + i} \)

- \( 1 - \prod_{i=1}^{h} \left( 1 - \frac{|A| + 1}{|A| + |B| + i} \right) \geq 1 - \left( 1 - \frac{|A|}{|A| + |B| + |C|} \right)^{|C|} \)
Point exists with probability at least

$$1 - \left( 1 - \frac{|A|}{|P|} \right)^{|C|}$$
Probabilistic Results

- $|P| = \alpha n \log_2 n$
- $|A| := |C| := \alpha \frac{n}{2}$ and $|B| := 2\alpha \frac{n}{2} \log_2 \frac{n}{2}$
- Point exists with probability at least

$$1 - \left(1 - \frac{1}{2 \log_2 n}\right)^{\alpha \frac{n}{2}} \geq 1 - \left(\frac{1}{e}\right)^{\alpha \frac{n}{4 \log_2 n}} \geq 1 - \left(\frac{1}{e}\right)^{\frac{\alpha \ln 2}{2}}$$

since $(1 - \frac{1}{x})^x \leq \frac{1}{e}$ on $[1, \infty)$ and $\frac{x}{\log x} \geq e$ on $(1, \infty)$

- ...
- $O(n \log^2 n)$, success with probability at least $\frac{1}{2}$
- Actually, $O(n \log n (\log \log n)^2)$
\[(x + 1)^\varepsilon - x^\varepsilon = \varepsilon x^{\varepsilon - 1} \geq \varepsilon x^{-1} \geq \varepsilon (x + 1)^{-1} \geq \varepsilon (2x)^{-1}\]

\[\alpha (2m) \log_2 \varepsilon (2m) + 8m - 4 \geq \alpha 2m \log_2 \varepsilon (2m) + 8m - 4\]

\[= \alpha 2m (\log_2 m + 1)^\varepsilon + 8m - 4\]

\[\geq \alpha 2m \left( \log_2 m + \frac{\varepsilon}{\log_2 m} \right) + 8m - 4\]

\[= \alpha 2m \log_2 m + \alpha 2 \frac{\varepsilon m}{\log_2 m} + 8m - 4\]

\[= 2 (\alpha m \log_2 m + 4m - 4) + 2 \left( \frac{\alpha \varepsilon m}{\log_2 m} \right) + 4\]

\[\geq 2 \left[ \alpha m \log_2 m + 4m - 4 \right] + 2 \left[ \frac{\alpha \varepsilon m}{\log_2 m} \right]\]
Probabilistic Results

- $\alpha := \frac{C \log_2 n}{\varepsilon^2}$ leads to
  \[
  \frac{C}{\varepsilon^2} n \log_2^{1+\varepsilon} n + O(n)
  \]

- $\varepsilon := \frac{1}{\log_2 \log_2 n}$ leads to
  \[
  Cn \log_2 n (\log_2 \log_2 n)^2 + O(n)
  \]

- where $C := \frac{80}{e^3 (\ln 2)^2} = 8.290 \cdots$
Probabilistic Results

Manfred Scheucher
Graz, June 18, 2015

Many cases ...
Figure: Tree in Case 3a
Probabilistic Results

Figure: Embedding in Case 3a

\[ f(n'_1) + f(n'_2) + f(n'_3) + f(n_2) + f(n_3) + 2f(n_4) + 4\alpha n. \]
Probabilistic Results

- For 4-Trees: $O(n^{\gamma_0 + \varepsilon})$
- $\gamma_0 := 1.3319 \cdots$ unique solution of the equation
  \[
  \left(\frac{1}{2}\right)^\gamma + \left(\frac{1}{24}\right)^\gamma + 2 \left(\frac{1}{3}\right)^\gamma + 2 \left(\frac{1}{8}\right)^\gamma = 1
  \]
- Let $\gamma > \gamma_0$ and let $\delta_\gamma = \frac{1}{24\gamma_0} - \frac{1}{24\gamma}$. Then $f_\gamma(x) = x^\gamma$ fulfills

  1. $f_\gamma(x) \geq f_\gamma\left(\frac{1}{2}x\right) + f_\gamma\left(\frac{1}{2}x\right) + \delta_\gamma x$,
  2. $f_\gamma(x) \geq f_\gamma\left(\frac{1}{2}x\right) + f_\gamma\left(\frac{1}{6}x\right) + 2f_\gamma\left(\frac{1}{3}x\right) + \delta_\gamma x$,
  3. $f_\gamma(x) \geq f_\gamma\left(\frac{1}{2}x\right) + f_\gamma\left(\frac{3}{8}x\right) + 2f_\gamma\left(\frac{1}{8}x\right) + \delta_\gamma x$, and
  4. $f_\gamma(x) \geq f_\gamma\left(\frac{1}{2}x\right) + f_\gamma\left(\frac{1}{24}x\right) + 2f_\gamma\left(\frac{1}{3}x\right) + 2f_\gamma\left(\frac{1}{8}x\right) + \delta_\gamma x$
Embedding Caterpillars - Basic Idea

Embedding of ($a_1, \ldots, a_i$) in $P_1$

Embedding of ($a_{l+1}, \ldots, a_k$) in $P_2$

Embedding of ($a_1, \ldots, a_k$) in $P$
Proof of $f_{OC4}(n) \leq \frac{4}{3}n + O(1)$

- $(\rightarrow 2 \rightarrow)$ admits a planar orthogeodesic embedding in any point set $P$ of size 3
Proof of $f_{OC4}(n) \leq \frac{4}{3}n + O(1)$

- $\rightarrow 3 \rightarrow$ admits a planar orthogeodesic embedding in any point set $P$ of size 4
Proof of $f_{OC4}(n) \leq \frac{4}{3} n + O(1)$

- $(\rightarrow 4 \rightarrow)$ admits a planar orthogeodesic embedding in any point set $P$ of size 6
Results for the General Case

<table>
<thead>
<tr>
<th></th>
<th>Planar L-Shaped</th>
<th>Nonplanar L-Shaped</th>
<th>Planar Orthogeodesic</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-Cat.</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>3-Tree</td>
<td>$n^2 - 2n + 2$</td>
<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>4-Cat.</td>
<td>$3n - 2$</td>
<td>$n + 1$</td>
<td>$[1.5n]$</td>
</tr>
<tr>
<td>4-Tree</td>
<td>$n^2 - 2n + 2$</td>
<td>$4n - 3$</td>
<td>$4n$</td>
</tr>
</tbody>
</table>

Table: Upper bounds given by Giacomo et al.

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<tr>
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<th>Planar Orthogeodesic</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-Tree</td>
<td>$0.334n^{1.585} + \mathcal{O}(n)$</td>
<td>$n$ [Giacomo et al.]</td>
<td>$n$ [Giacomo et al.]</td>
</tr>
<tr>
<td>4-Cat.</td>
<td>$1.334n + \mathcal{O}(1)$</td>
<td>$n$</td>
<td>$1.334n + \mathcal{O}(1)$</td>
</tr>
<tr>
<td>4-Tree</td>
<td>$0.339n^{1.585} + \mathcal{O}(n)$</td>
<td>$2.334n + \mathcal{O}(1)$</td>
<td>$1.5n + \mathcal{O}(1)$</td>
</tr>
</tbody>
</table>

Table: Best upper bounds currently known
Burnside’s Lemma

Notation:

- orbit \( \overline{x} := \{gx \mid g \in G\} \)
- fixed points \( X_g := \{x \mid gx = x\} \)
- stabilisers \( G_x := \{g \mid gx = x\} \)
Burnside’s Lemma

\[ \sum_{g \in G} |X_g| = \sum_{g \in G, x \in X, gx = x} 1 = \sum_{x \in X} |G_x| \]

- \( \bar{x} \simeq G/G_x \), because \( \phi_g : x \mapsto gx \) has image \( \phi_g(\bar{x}) = \bar{x} \), \( \phi_g = \phi_{gh} \) only for \( h \in G_x \) . . .

- \( |\bar{x}| = \frac{|G|}{|G_x|} \) (Lagrange’s Theorem)

\[ \# \text{orbits} = \sum_{x \in X} \frac{1}{|\bar{x}|} = \sum_{x \in X} \frac{|G_x|}{|G|} = \frac{1}{|G|} \sum_{g \in G} |X_g| \]
Burnside’s Lemma - An Example ($n = 4$)

- **Group actions:** Rotate + Mirror
  
  \[ G = \{ r_0, r_{90}, r_{180}, r_{270}, m_0, m_{90}, m_{180}, m_{270} \} \]

- $r_0 = id$: all $4! = 24$ elements are left unchanged
- $r_{90}$ and $r_{270}$: 2 each
- $r_{180}$: 8

![Fixed points for $r_{90}$ (left) and $r_{180}$ (right).](image-url)

**Figure:** Fixed points for $r_{90}$ (left) and $r_{180}$ (right).
Burnside’s Lemma - An Example ($n = 4$)

- $m_0$: 10
- $m_{90}$ and $m_{270}$: 0 (not possible)
- $m_{180}$: 10 (analogous to $m_0$)

Figure: Fixed points for $m_0$ (left) and $m_{90}$ (right).
Burnside's Lemma - An Example ($n = 4$)

- $\#orbits(4) = \frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{24+2+8+2+10+0+10+0}{8} = 7$
- $\#orbits(n) \geq \frac{n!}{8}$
- $1, 1, 2, 7, 23, 115, 694, 5282, 46066, 456454, \ldots$  
  https://oeis.org/A000903

Figure: General point sets of size 4 (+symmetry).
Thank you for your attention!