A superlinear lower bound on the number of 5-holes

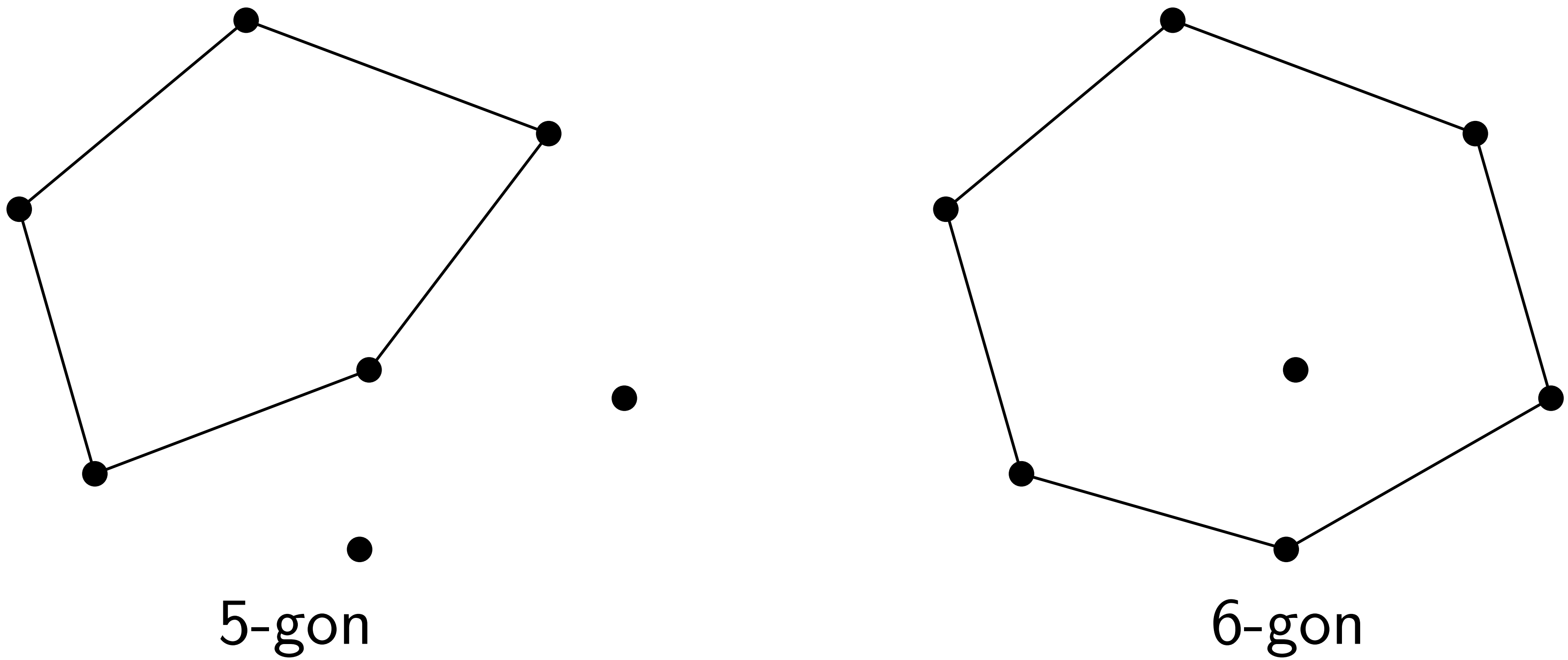
Oswin Aichholzer[1], Martin Balko[2,3], Thomas Hackl[1], Jan Kynčl[2], Irene Parada[1], Manfred Scheucher[1,3], Pavel Valtr[2,3], and Birgit Vogtenhuber[1]

[1] Graz University of Technology
[2] Charles University, Prague
[3] Alfréd Rényi Institute of Mathematics, Budapest

Introduction: k-gons

a finite point set P in the plane is in *general position* if $\nexists$ collinear points in P

a *k-gon (in P)* is the vertex set of a convex k-gon



Introduction: k-gons

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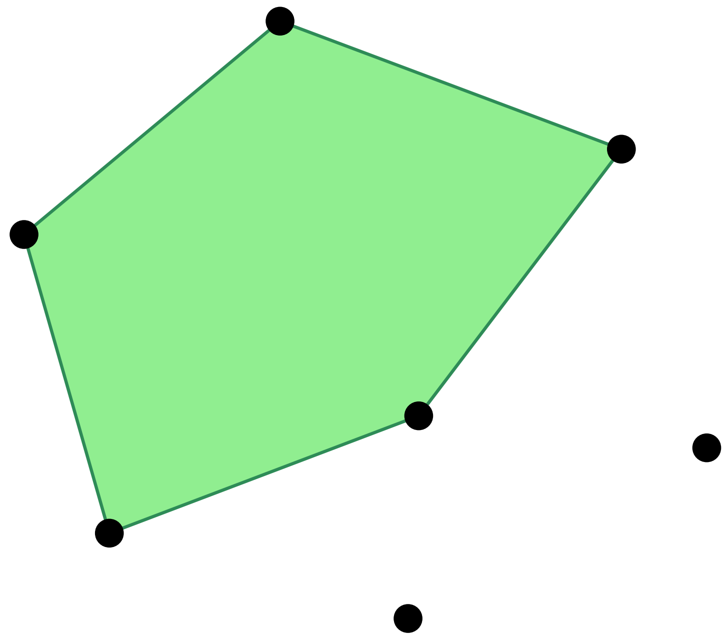
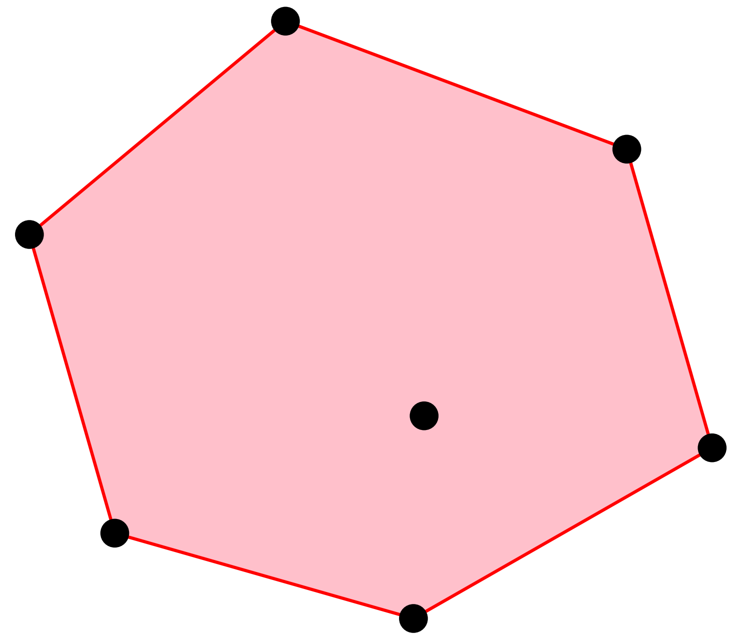
Theorem (Erdős and Szekeres '35).
For every $k \geq 3$, there is a smallest integer $n = n(k)$ such that every set of at least n points in general position contains a k-gon.

Introduction: k-holes

a *k-hole (in P)* is the vertex set of a convex k-gon containing no other points of P



5-hole



not a 6-hole

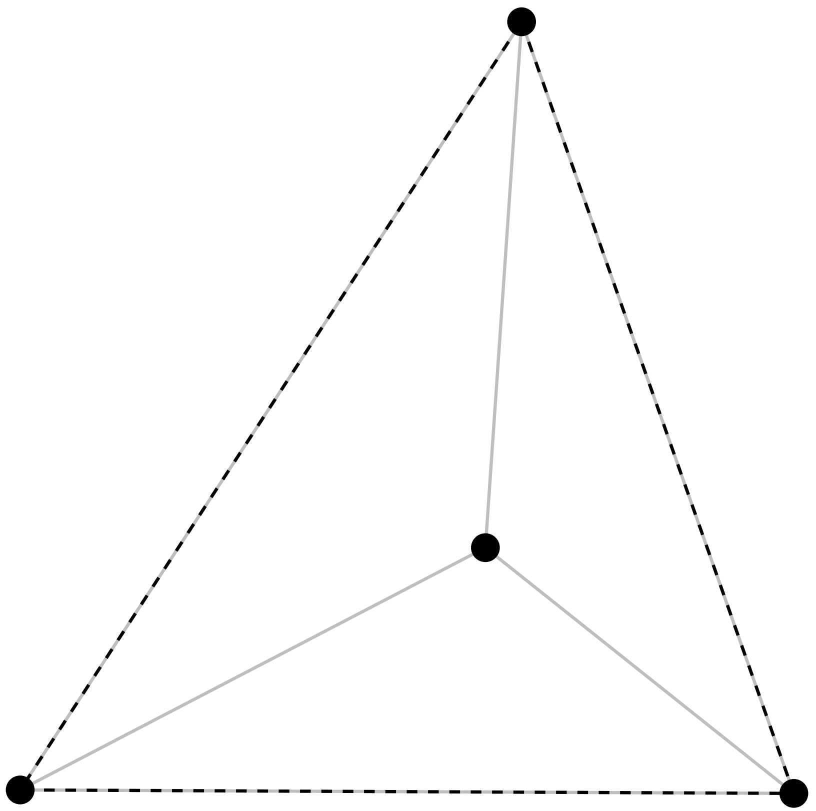
Introduction: k-holes

a *k-hole (in P)* is the vertex set of a convex k-gon containing no other points of P

$\forall k$, is there a k-hole in every sufficiently large point set? [Erdős, 1970's]

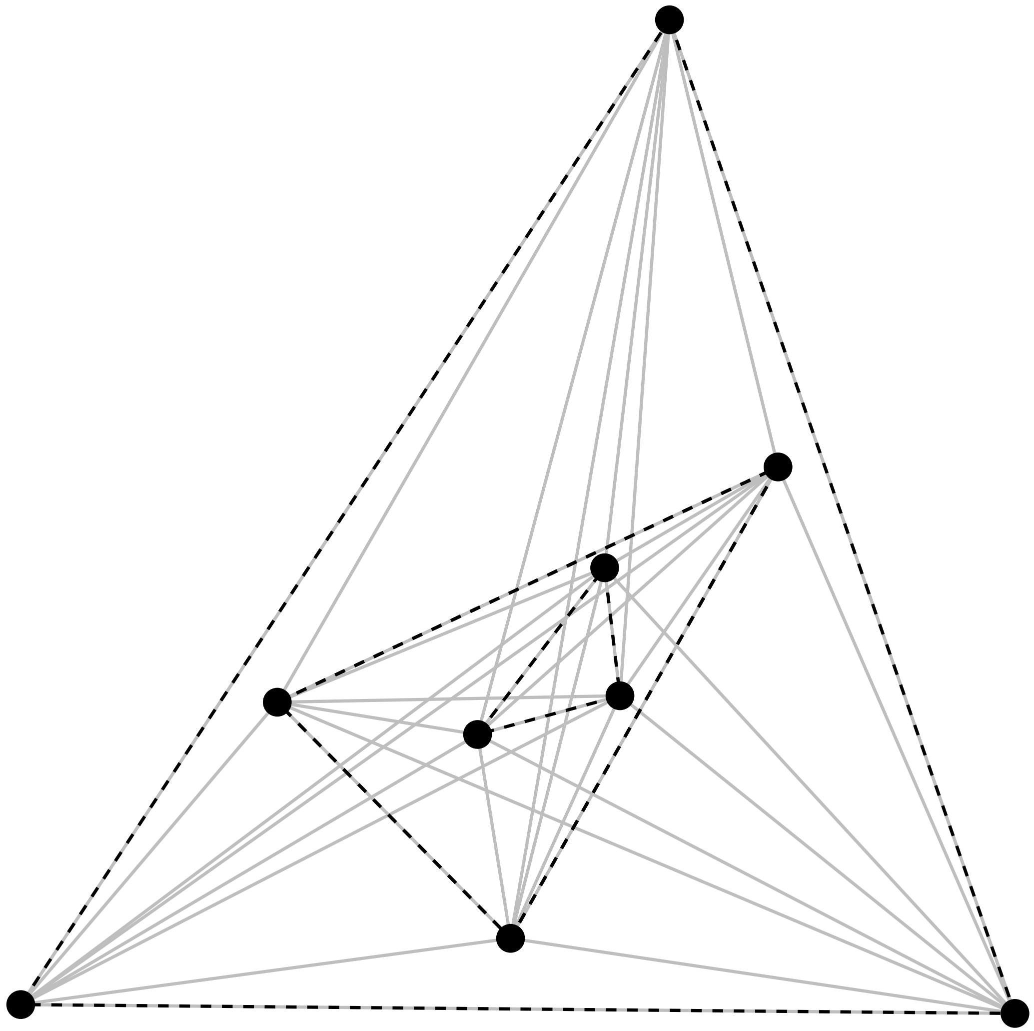
- 3 points $\Rightarrow$ $\exists$ 3-hole
- 5 points $\Rightarrow$ $\exists$ 4-hole
- 10 points $\Rightarrow$ $\exists$ 5-hole [Harborth '78]
- $\exists$ arbitrarily large point sets with no 7-hole [Horton '83]
- Sufficiently large point sets $\Rightarrow$ $\exists$ 6-hole [Gerken '08 and Nicolás '07, independently]

Introduction: k-holes



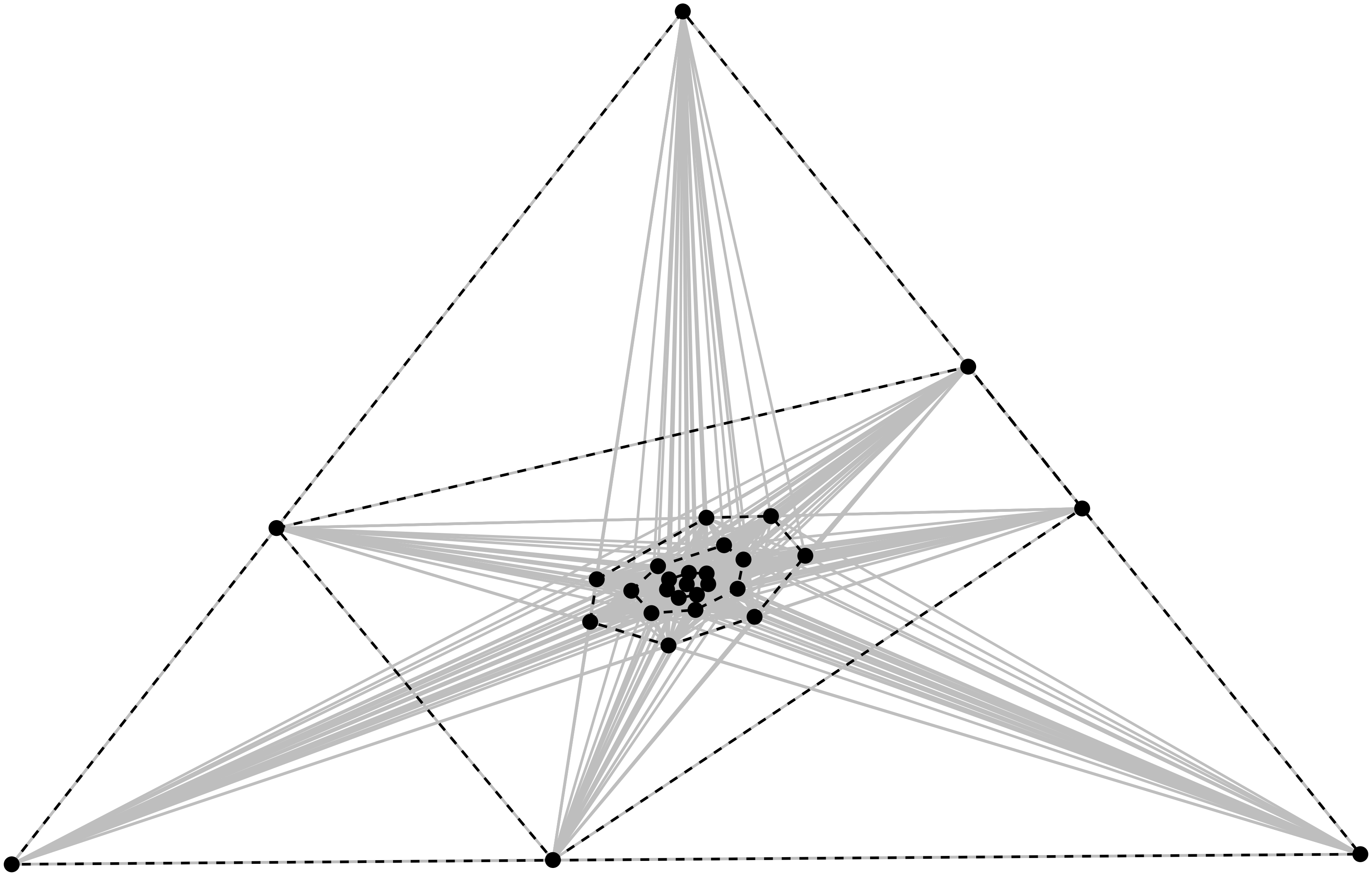
4 points, no 4-hole

Introduction: k-holes



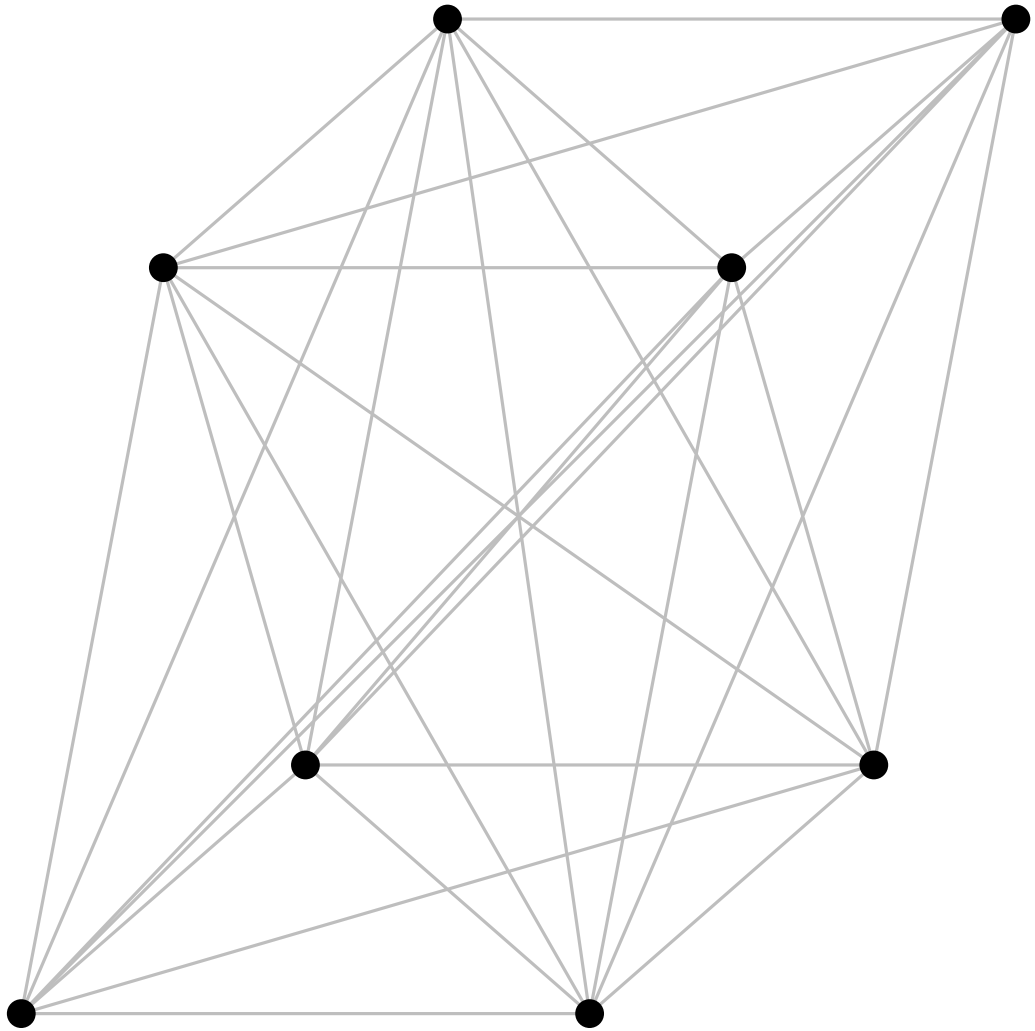
9 points, no 5-hole

Introduction: k-holes



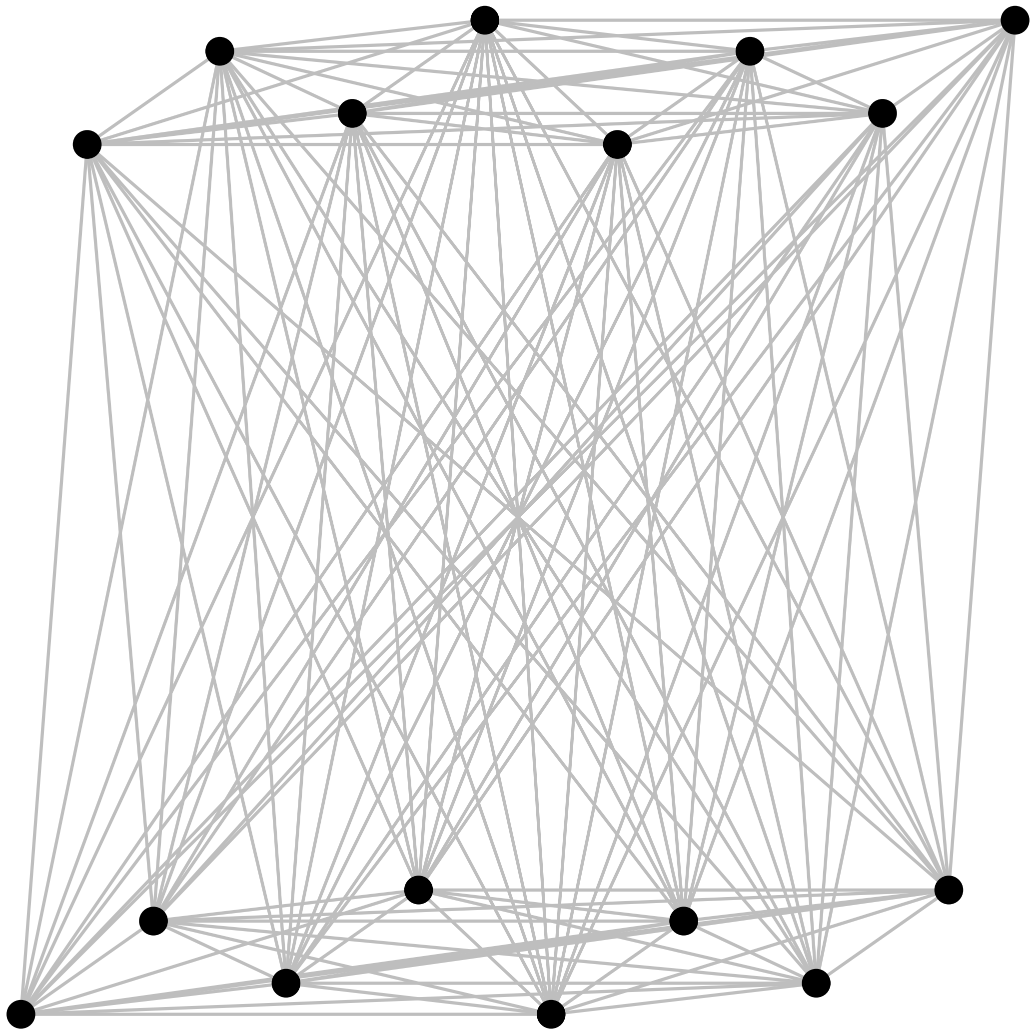
29 points, no 6-hole [Overmars '02]

Introduction: k-holes



Horton's construction for $n = 2^3$ points, no 7-holes

Introduction: k-holes



Horton's construction for $n = 2^4$ points, no 7-holes

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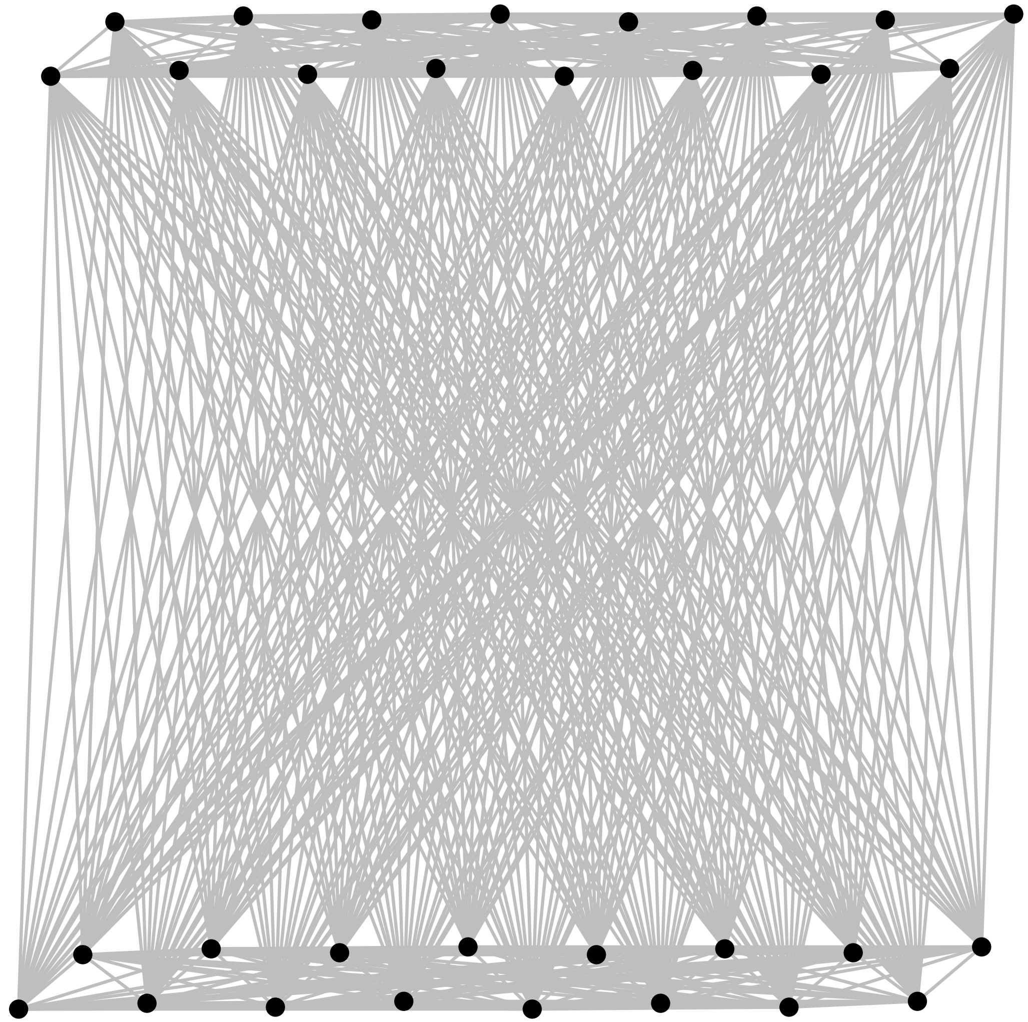

Horton's construction for $n = 2^5$ points, no 7-holes

Introduction: Asymptotic Bounds

$h_k(n)$ = minimum number of k-holes among all sets of n points in the plane in general position

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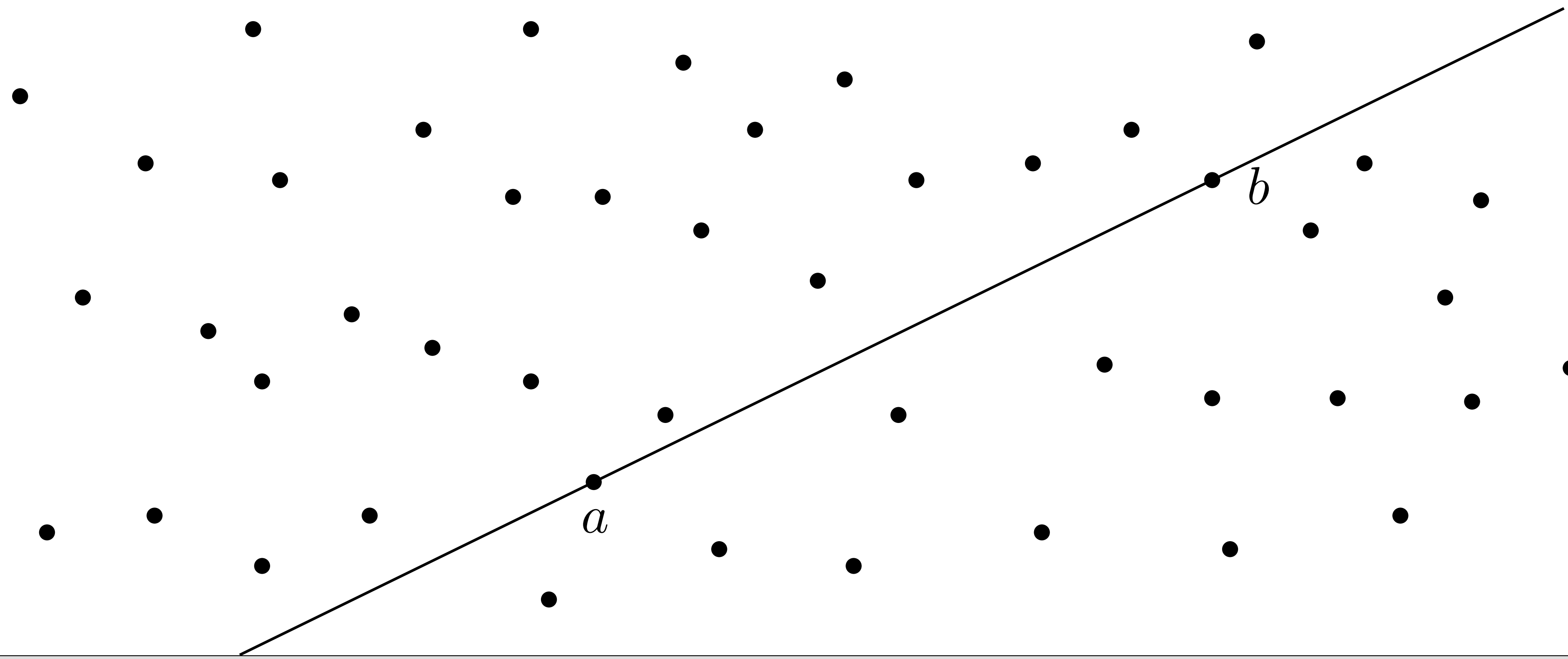

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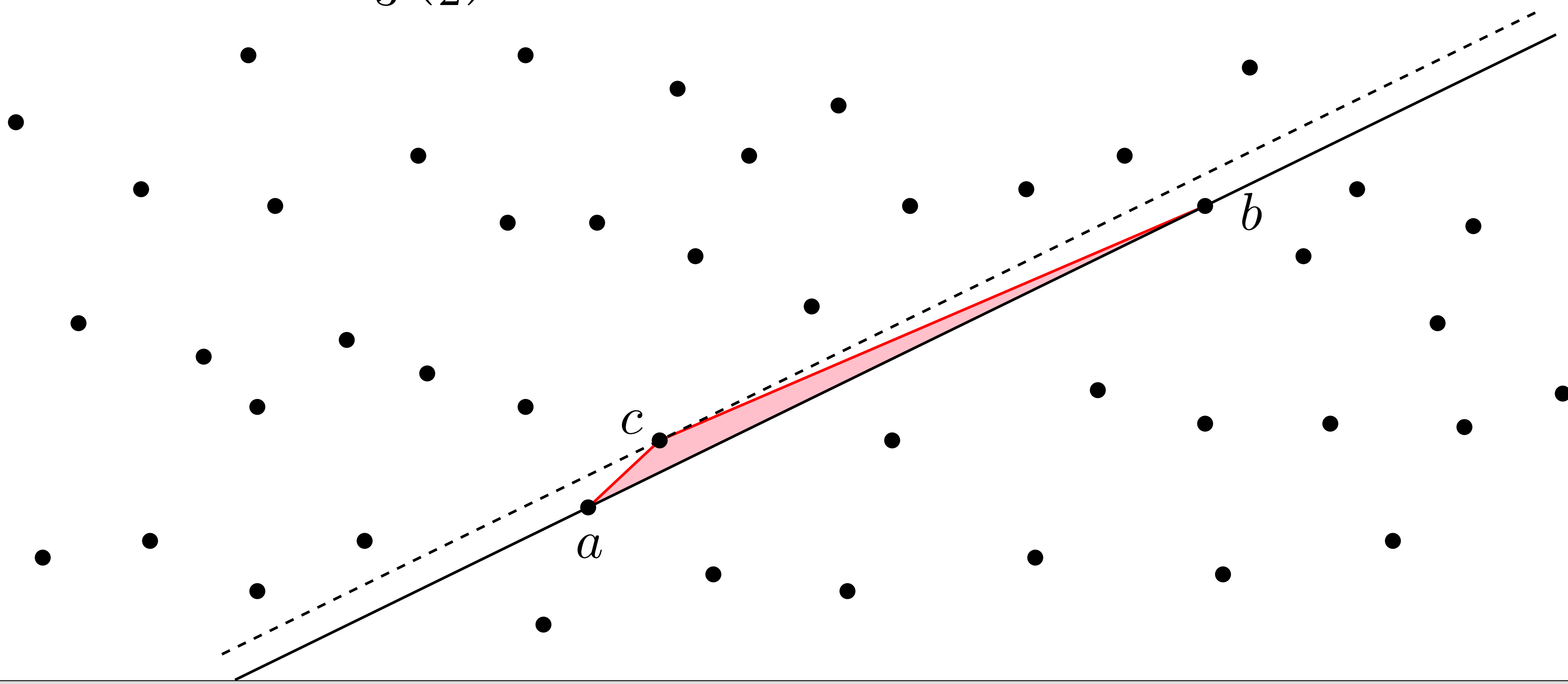

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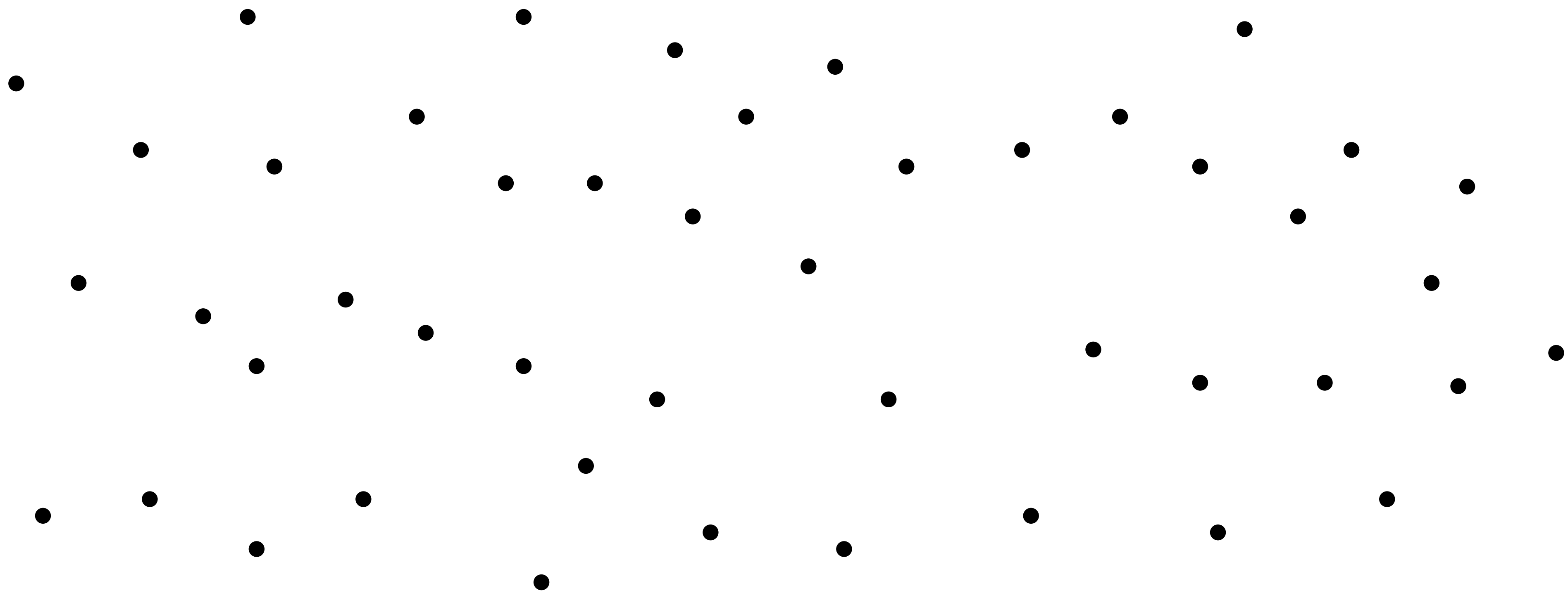

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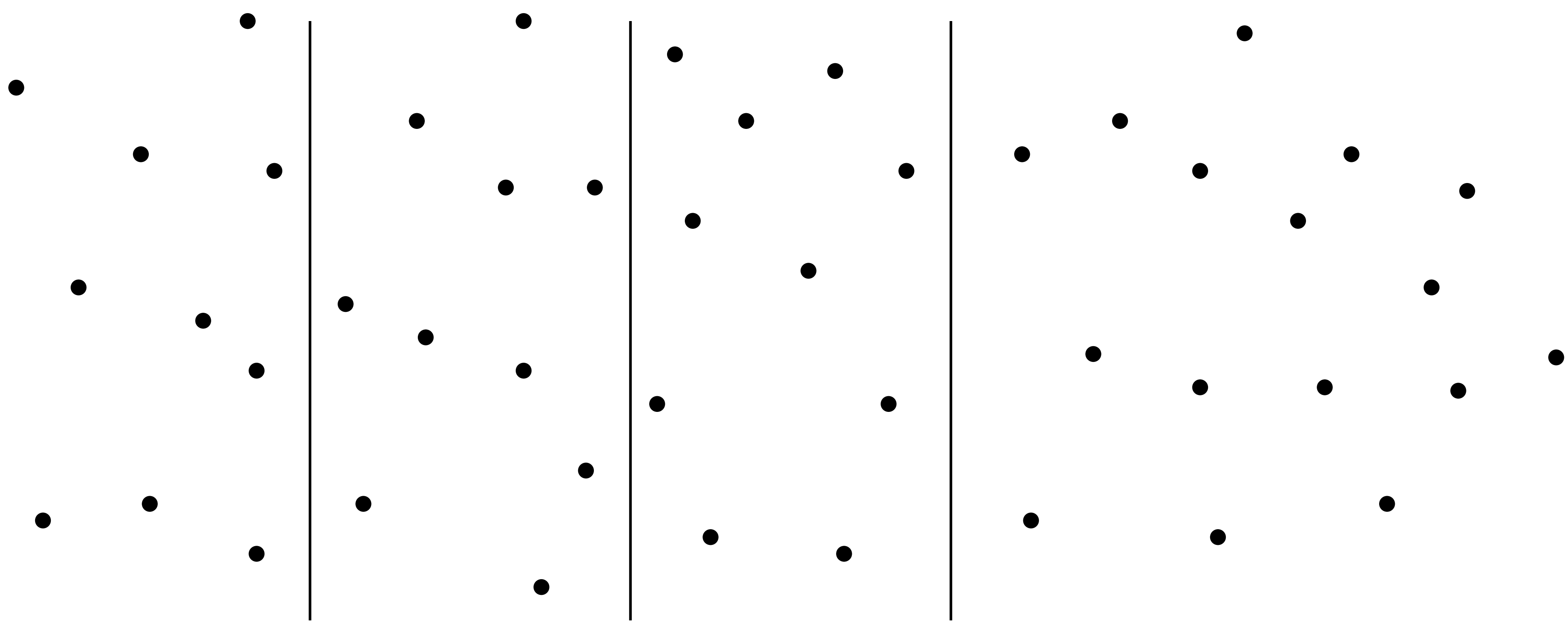

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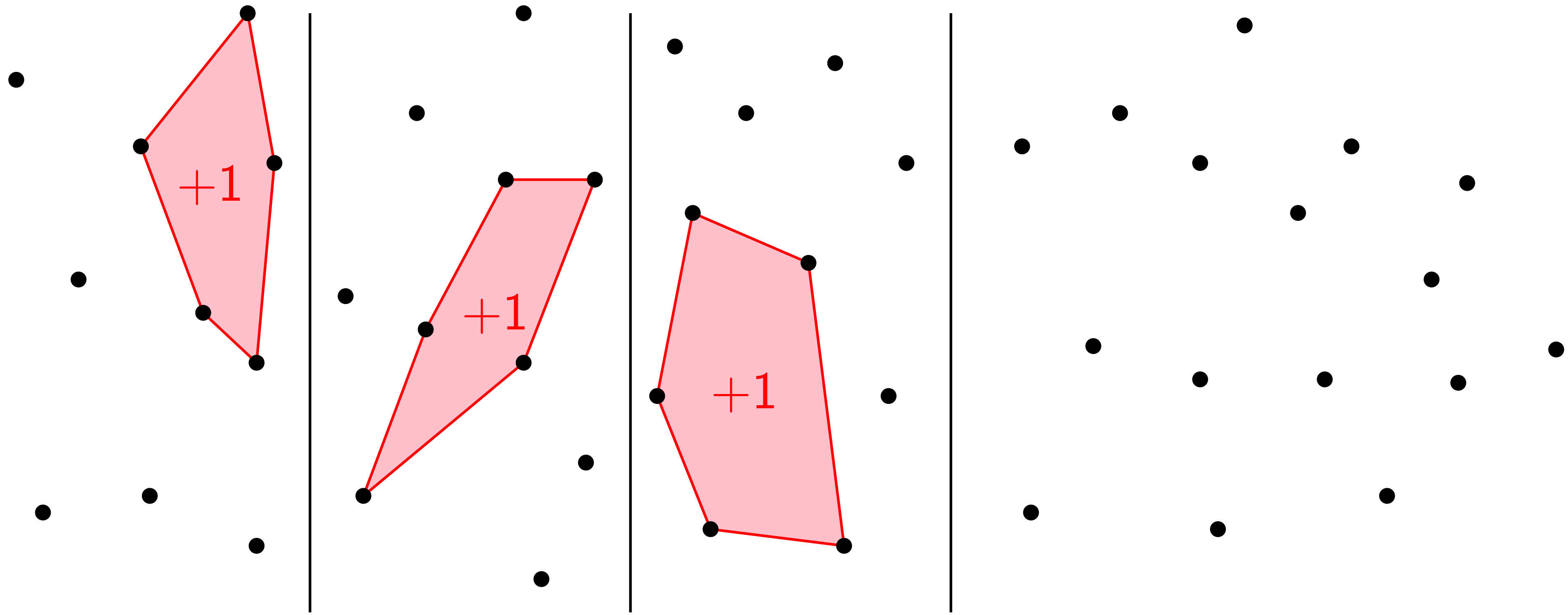

Introduction: Asymptotic Bounds

$h_k(n)$ = minimum number of k-holes among all sets of n points in the plane in general position

[Bárány and Füredi '87, Bárány and Valtr '04]

- $h_3(n)$ and $h_4(n)$ both $\Theta(n^2)$
- $h_k(n) = 0$ for $k \geq 7$ [Horton '83]
- $h_5(n)$ and $h_6(n)$ both $\Omega(n)$ and $O(n^2)$

[Harborth '78] [Gerken '08, Nicolás '07]

Introduction: Asymptotic Bounds

$h_k(n)$ = minimum number of k-holes among all sets of n points in the plane in general position

Conjecture 1: $h_5(n)$ is quadratic in n.

Conjecture 2: $h_5(n)$ is superlinear in n.

Introduction: 5-Holes

- $h_5(9) = 0$
- $h_5(10) = 1$
- $h_5(11) = 2$
- $h_5(12) = 3$

$\left.\begin{array}{l} h_5(9) = 0 \\ h_5(10) = 1 \end{array}\right]$ [Harborth '78]

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[Harborth '78]

[Dehnhardt '87]

$\longrightarrow h_5(n) \geq 3n/4 + o(n)$

[Aichholzer, Fabila-Monroy, Hackl, Huemer, Pilz, and Vogtenhuber '14]

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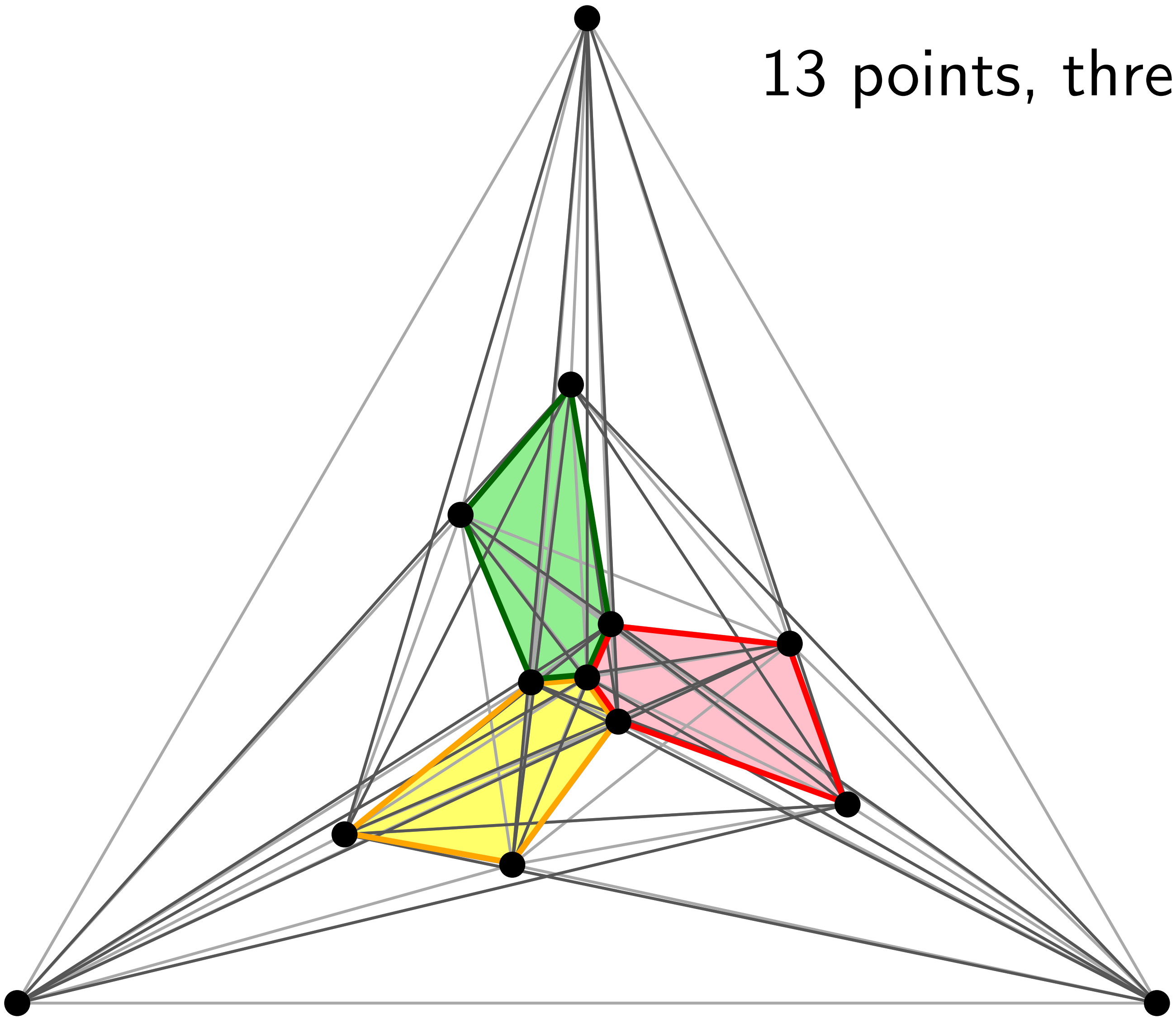
[Dehnhardt '87]

$h_5(n) \geq 3n/4 + o(n)$ [AFHHPV '14]

- $h_5(13) = 3$
- $h_5(14) = 6$
- $h_5(15) = 9$
- $h_5(16) \in \{10, 11\}$

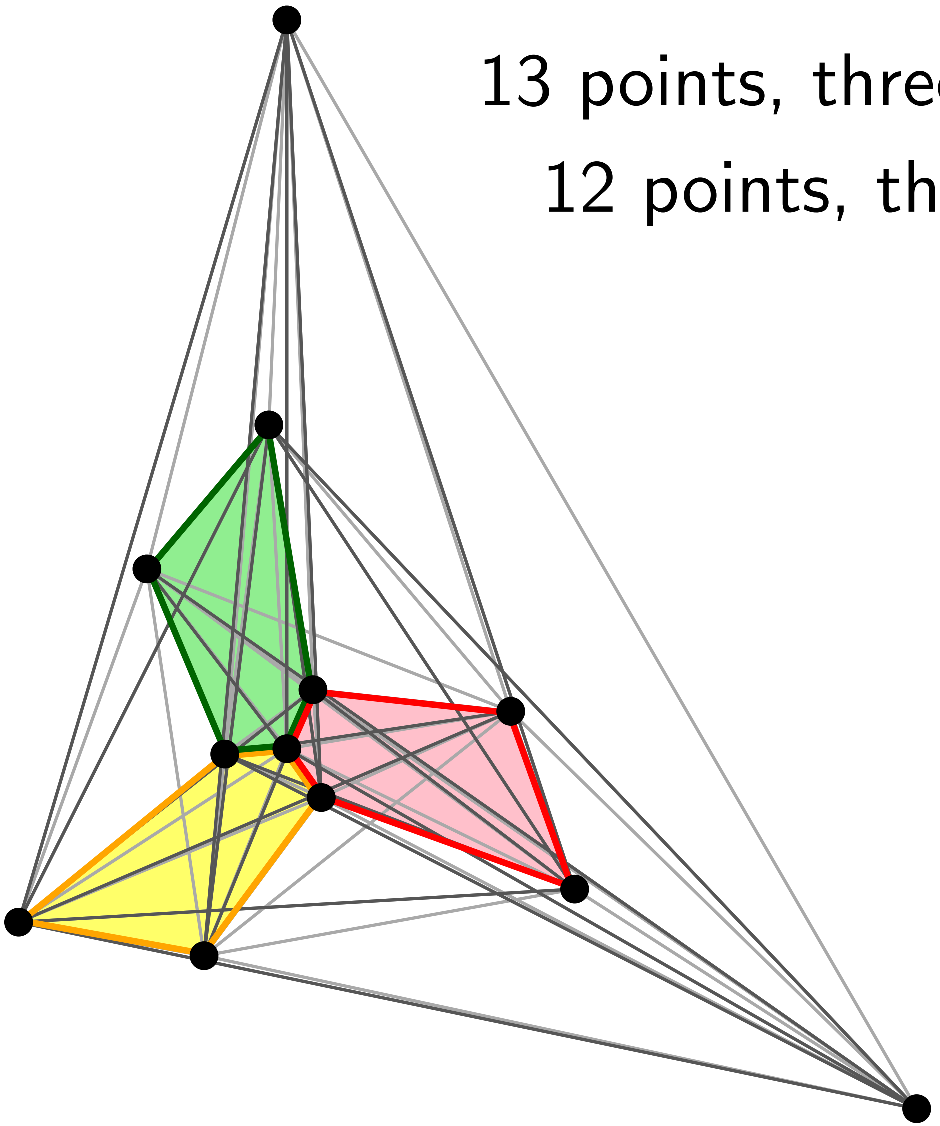
[Bachelor's thesis of S. '13]

Introduction: 5-Holes



13 points, three 5-holes [S. '13]

Introduction: 5-Holes



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12 points, three 5-holes

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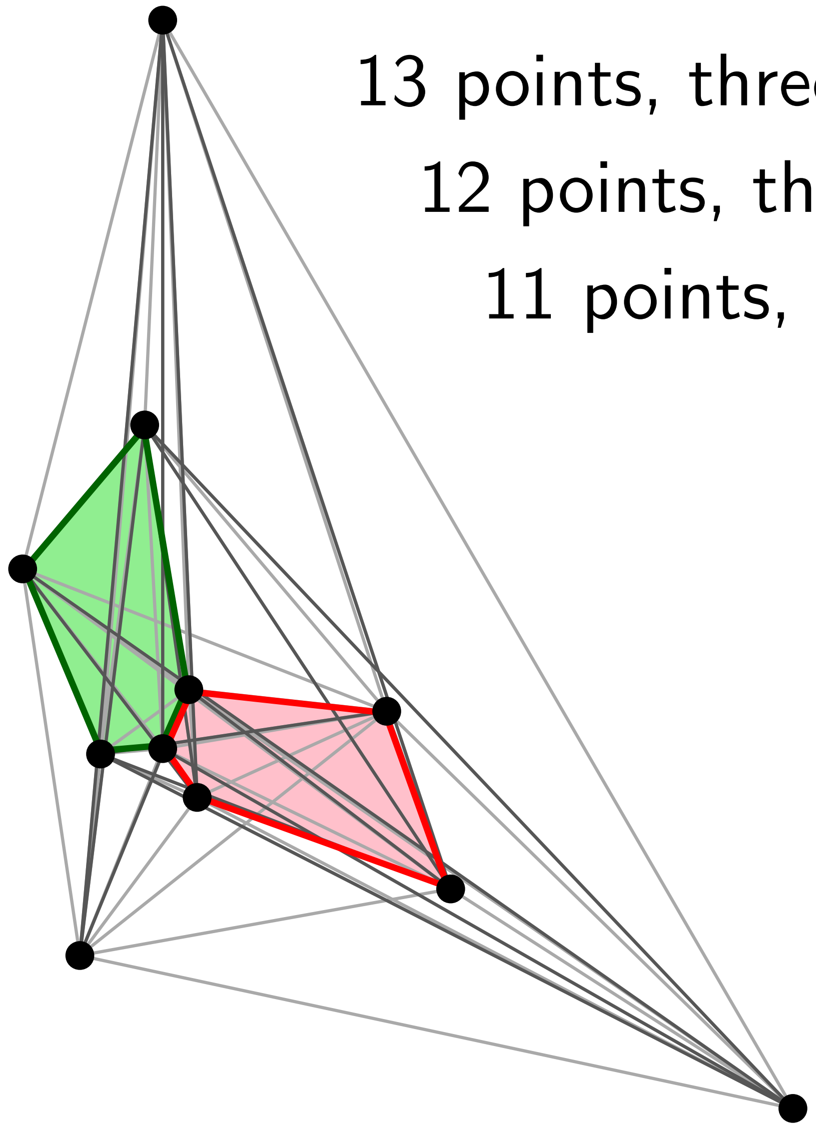

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12 points, three 5-holes

11 points, two 5-holes

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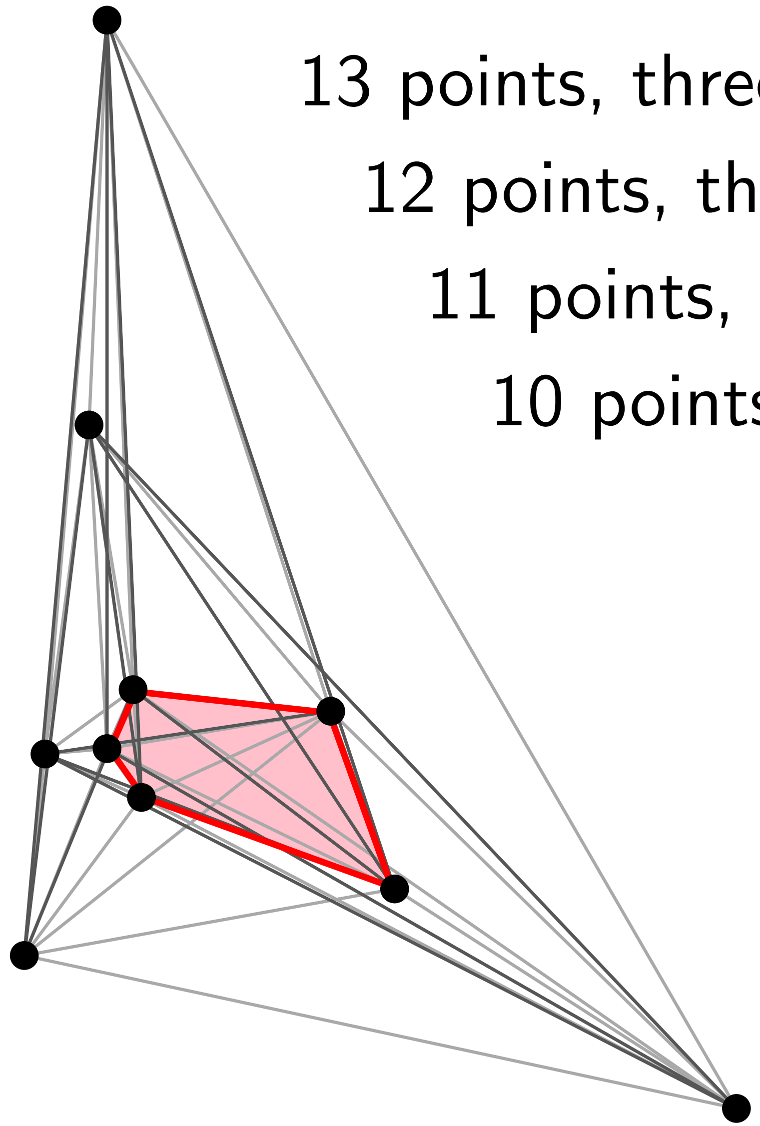

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11 points, two 5-holes

10 points, one 5-hole

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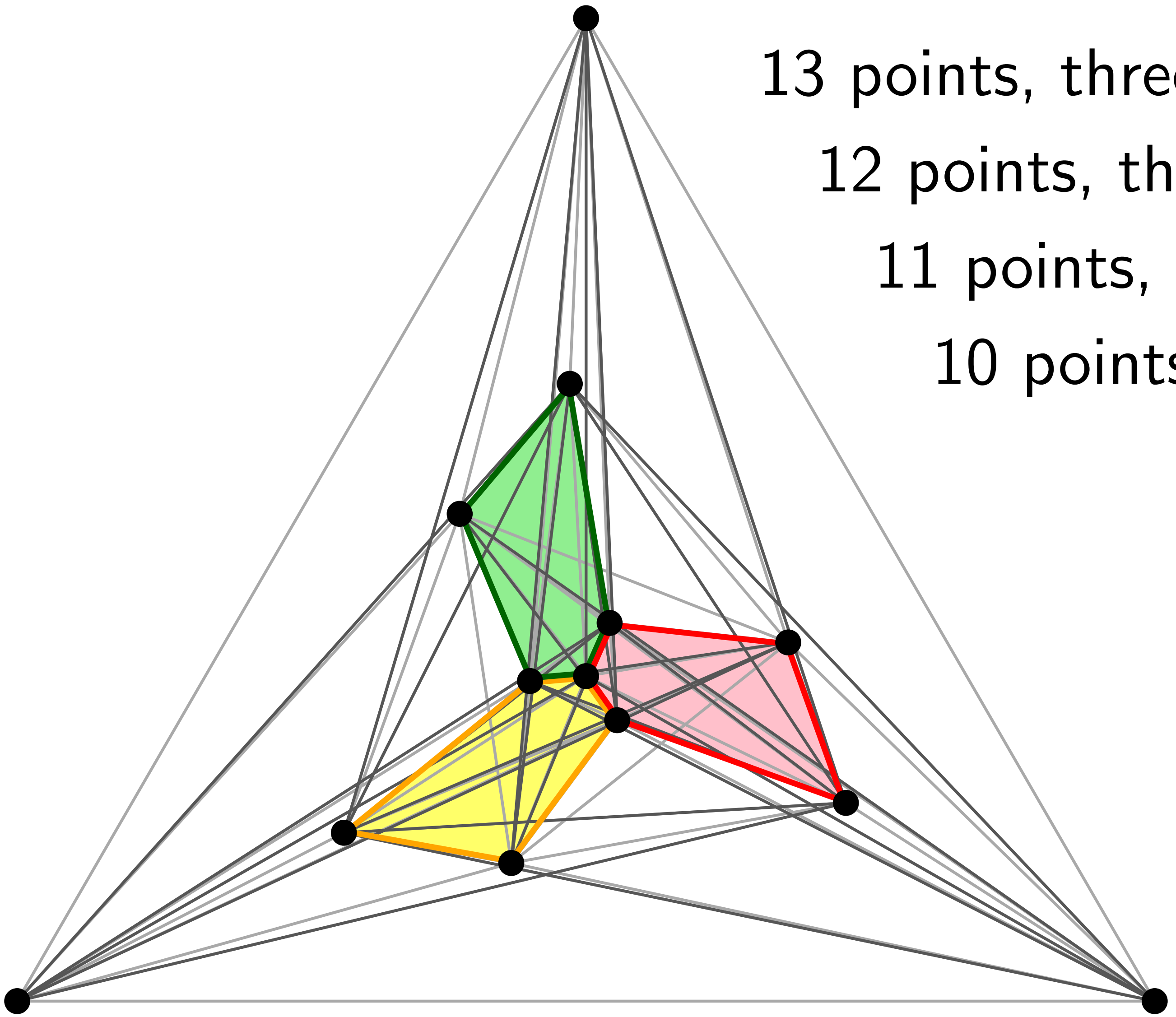

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Our Contribution

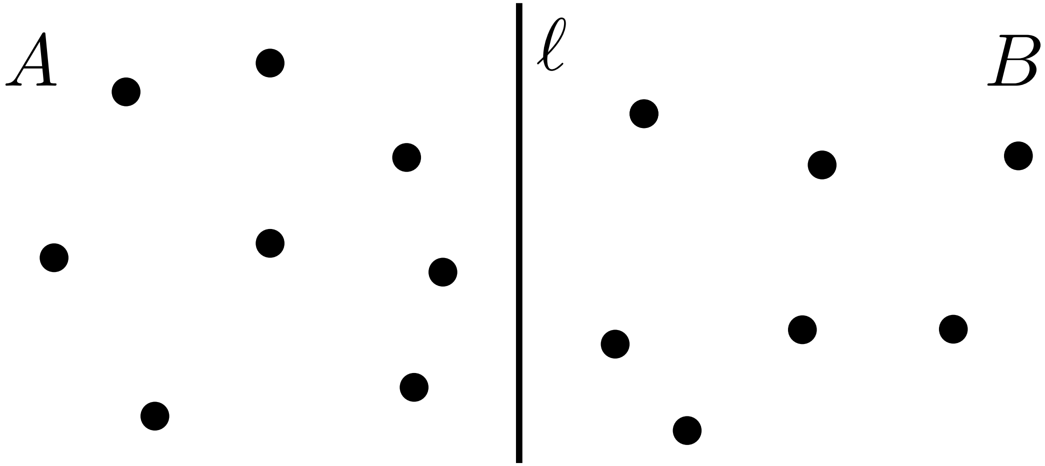
Theorem 1: There is a fixed constant $c > 0$ such that for every integer $n \geq 10$ we have $h_5(n) \geq cn \log^{4/5} n$.

This solves Conjecture 2.

Conjecture 1 is still open.

Our Contribution

$P = A \cup B$ is *ℓ-divided* if the line ℓ contains no point of P and partitions P into two non-empty subsets A and B



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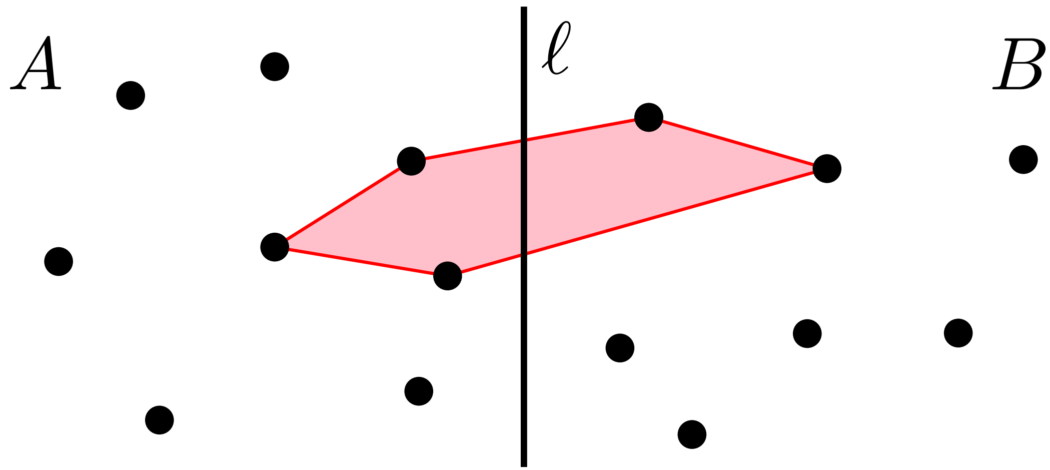
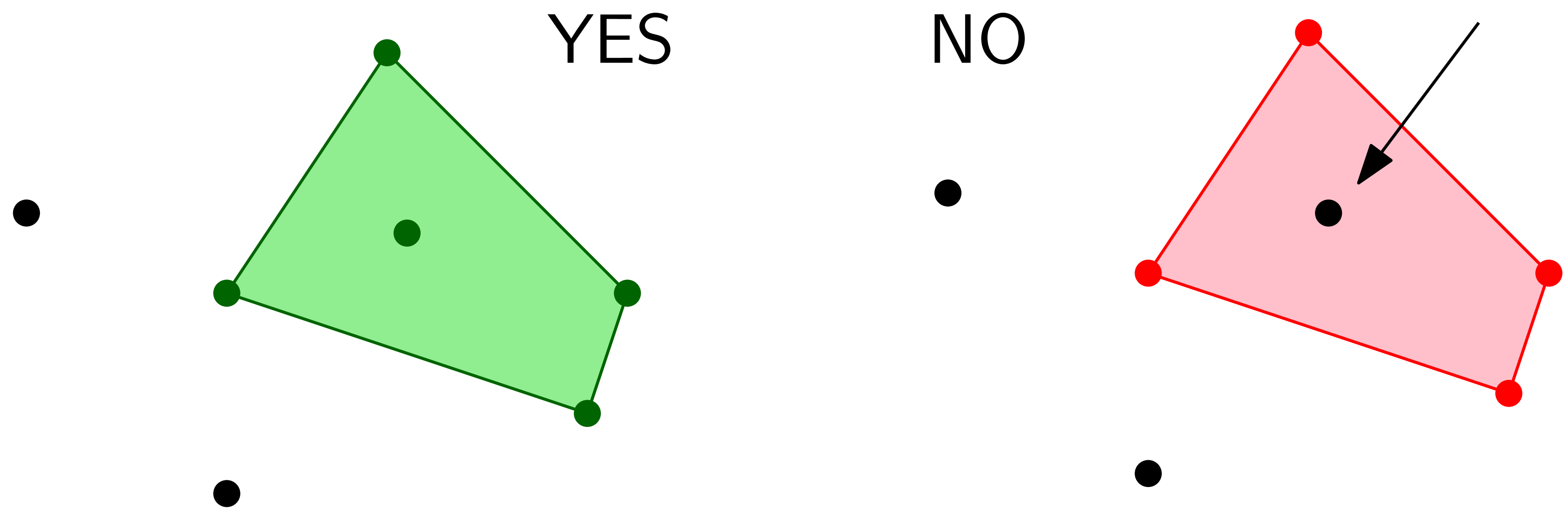


Theorem 2: Let $P = A \cup B$ be an ℓ-divided set with $|A|, |B| \geq 5$ and with neither A nor B in convex position. Then there is an ℓ-divided 5-hole in P.

The proof is computer-assisted.

Islands

an *island (of P)* is a subset Q of P with $P \cap \mathrm{conv}(Q) = Q$



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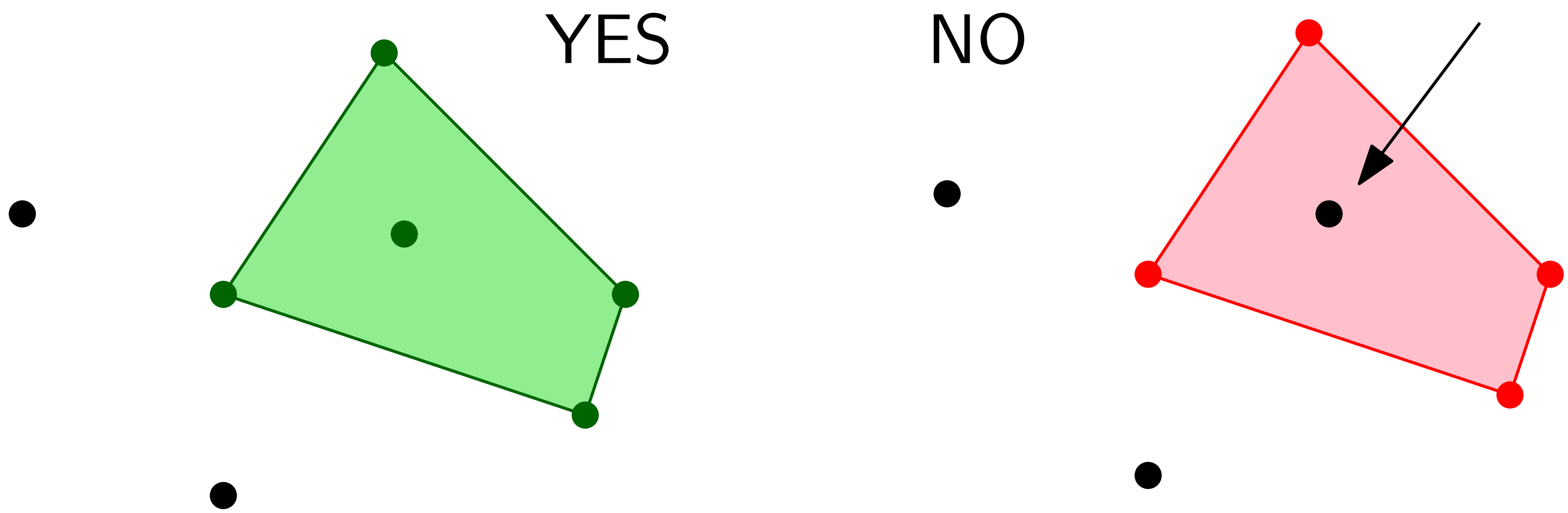


Observation: k-holes in an island of P are also k-holes in P

Proof of Theorem 1

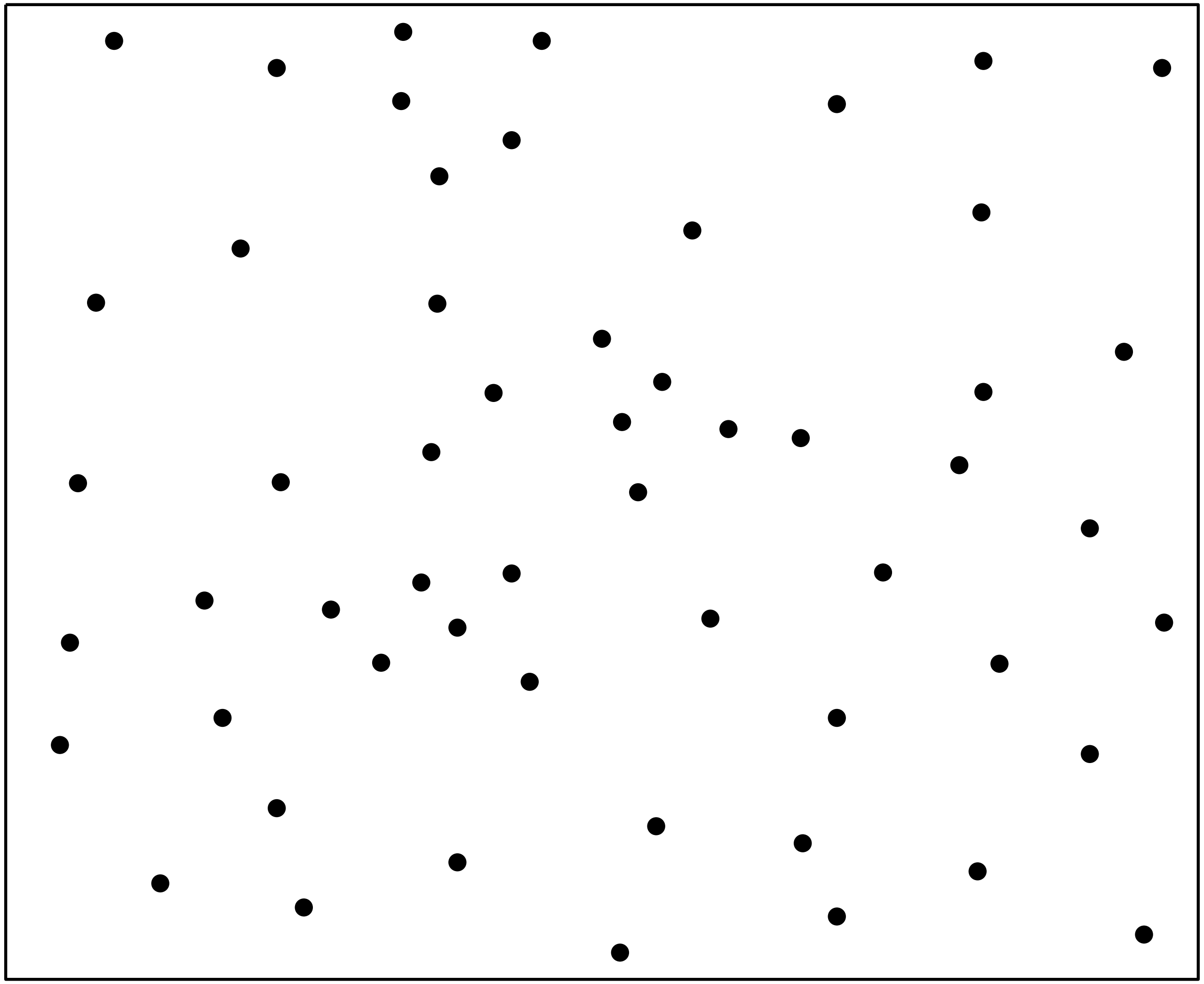
Theorem 1: $\exists\, c > 0$ s.t. $h_5(n) \geq cn \log^{4/5} n$ for $n \geq 10$.

Sketch of proof: We proceed by induction on $t = \log_2 n$.

Induction Base ($t = 5^5$): We have $n = 2^t > 10$.
Since $h_5(n) \geq h_5(10) = 1$, there are at least $c \cdot n \log_2^{4/5} n$ 5-holes in P for c small enough.

Proof of Theorem 1

Induction Step ($t > 5^5$):



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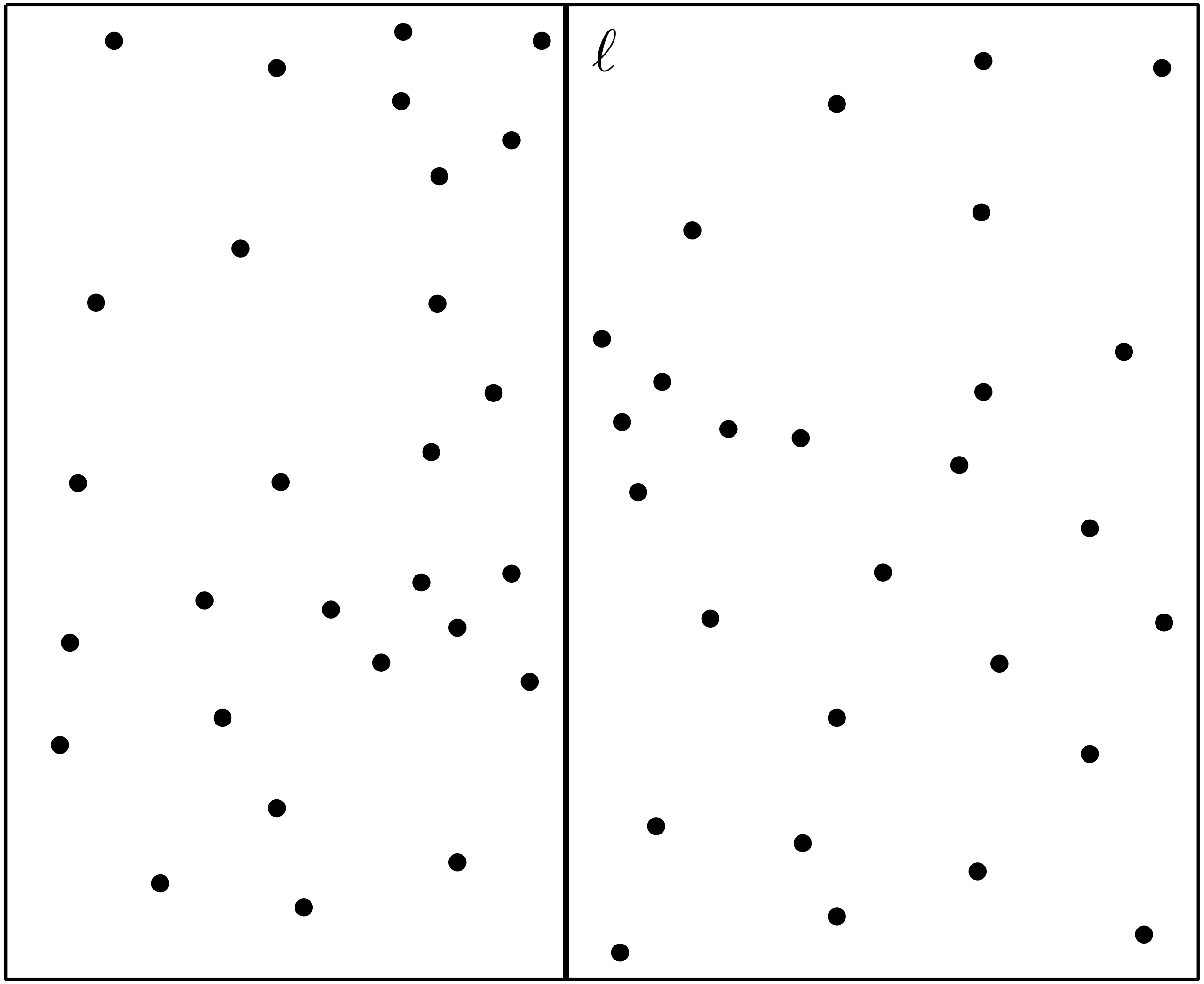

We partition

$$P = A \cup B$$

such that

$$|A| = \frac{n}{2} = |B|$$

Proof of Theorem 1

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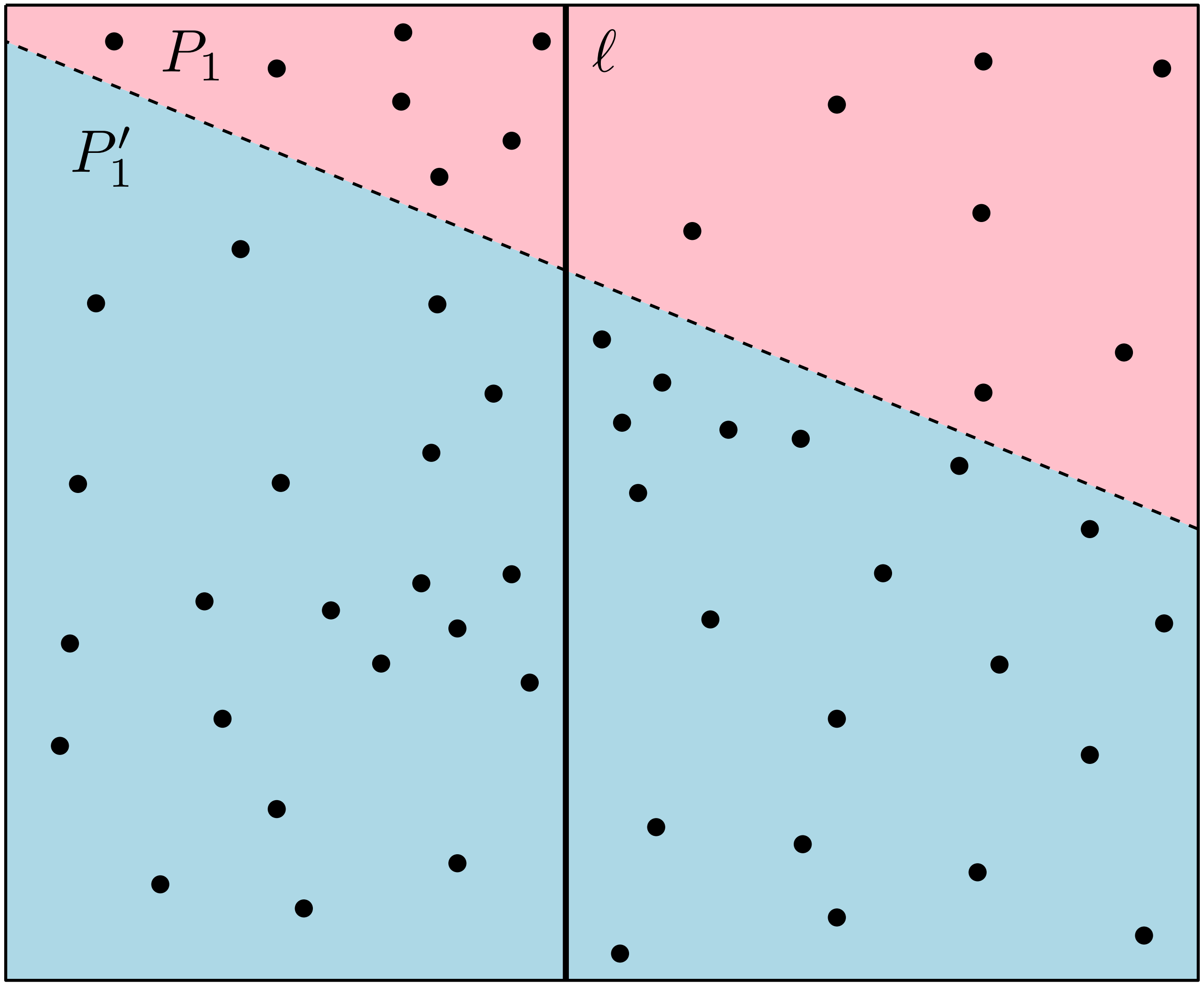

We partition

$$P = P_1 \cup P_1'$$

such that

$$|A \cap P_1| = r$$

$$|B \cap P_1| = r$$

Proof of Theorem 1

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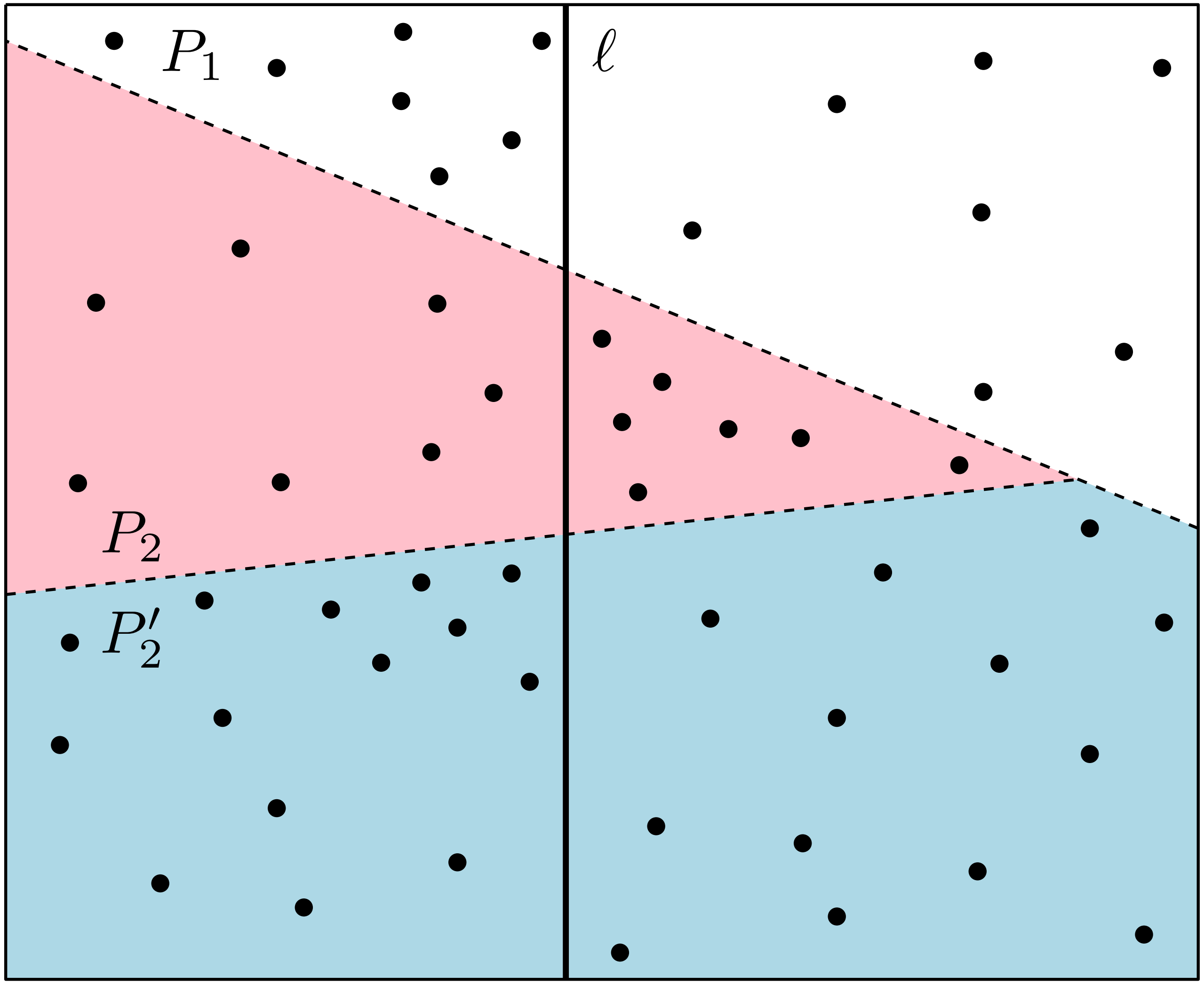


We partition

$$P_1 = P_2 \cup P_2'$$

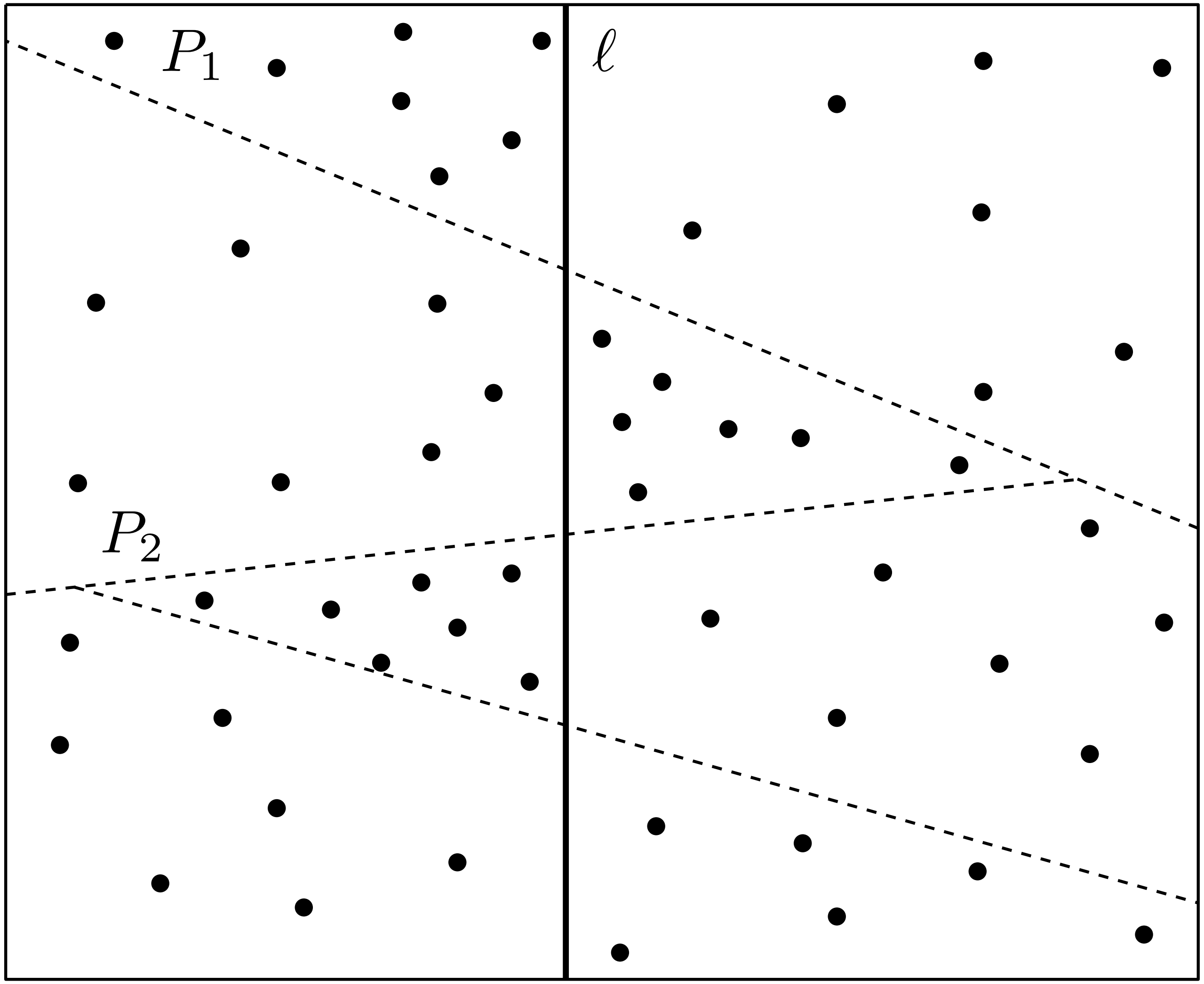
such that

$$|A \cap P_2| = r$$

$$|B \cap P_2| = r$$

Proof of Theorem 1

Induction Step ($t > 5^5$):



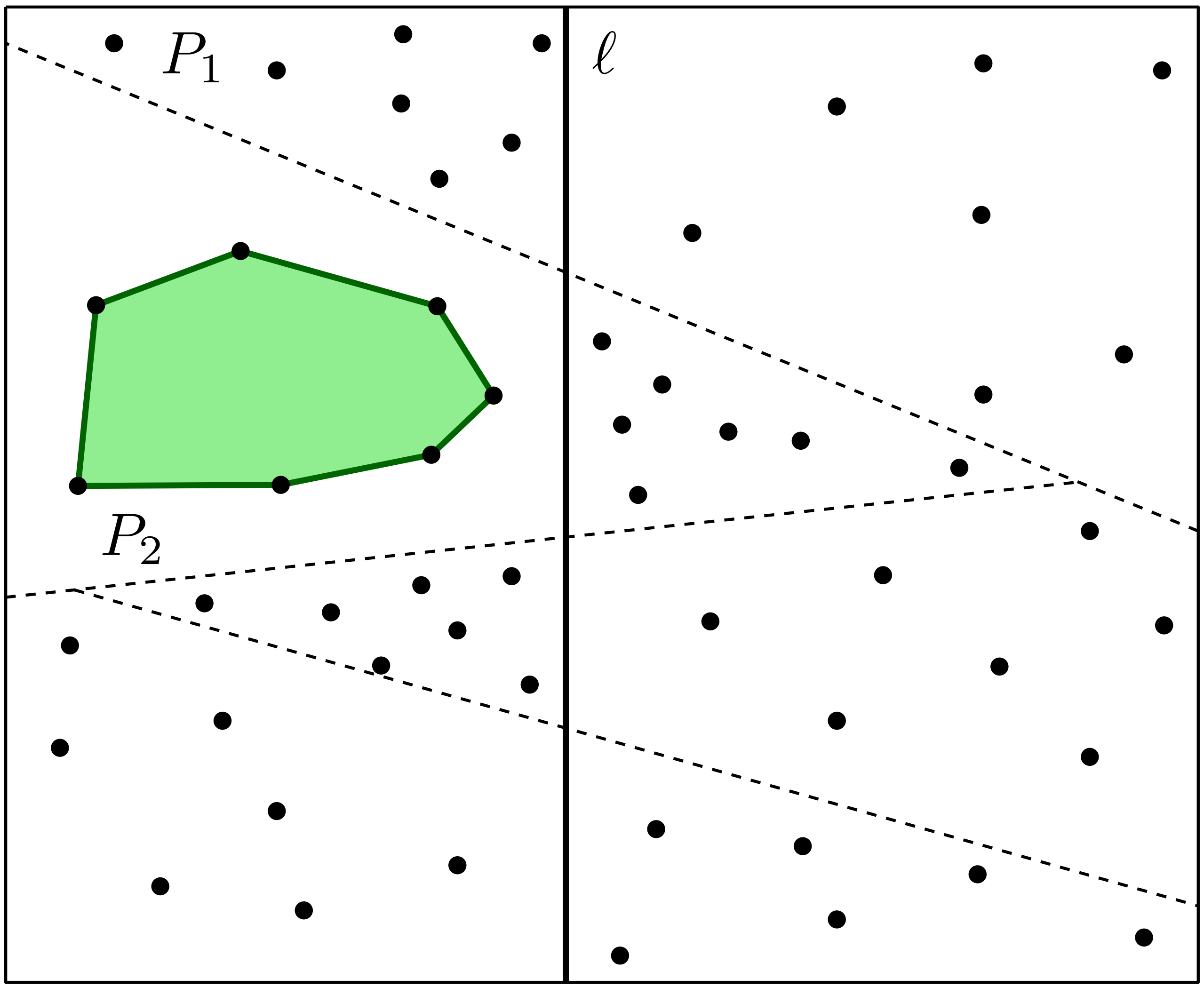
and so on...

$\Rightarrow s = \frac{n}{2r}$ islands

$P_1, \ldots, P_s$

Proof of Theorem 1

Induction Step ($t > 5^5$):



$P_i \cap A$ convex or
$P_i \cap B$ convex

$\Rightarrow \binom{r}{5}$ 5-holes

Proof of Theorem 1

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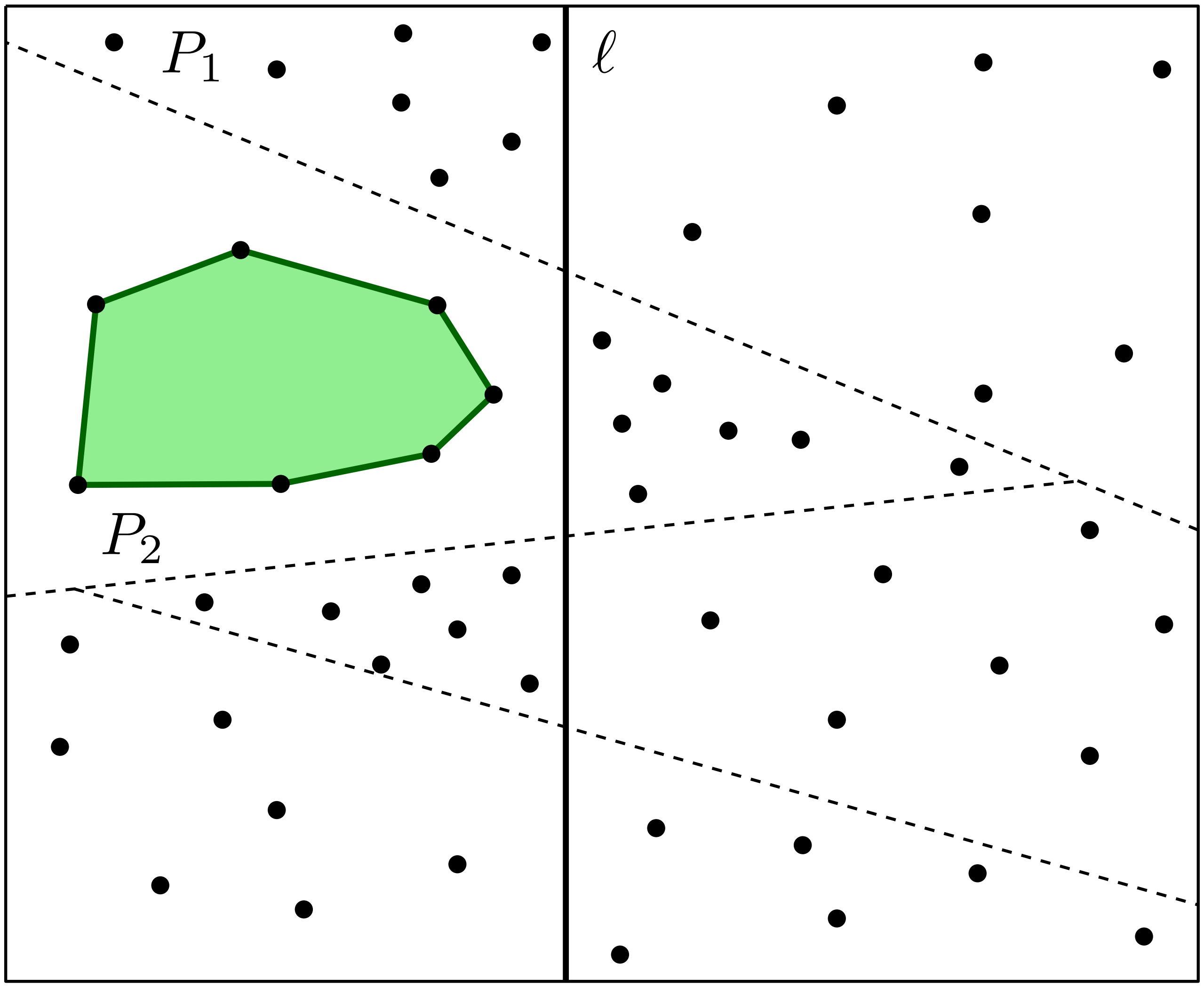

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If this is the case for $\frac{s}{2}$ of the islands P_i, we count at least $\frac{s}{2}\binom{r}{5}$ 5-holes

Proof of Theorem 1

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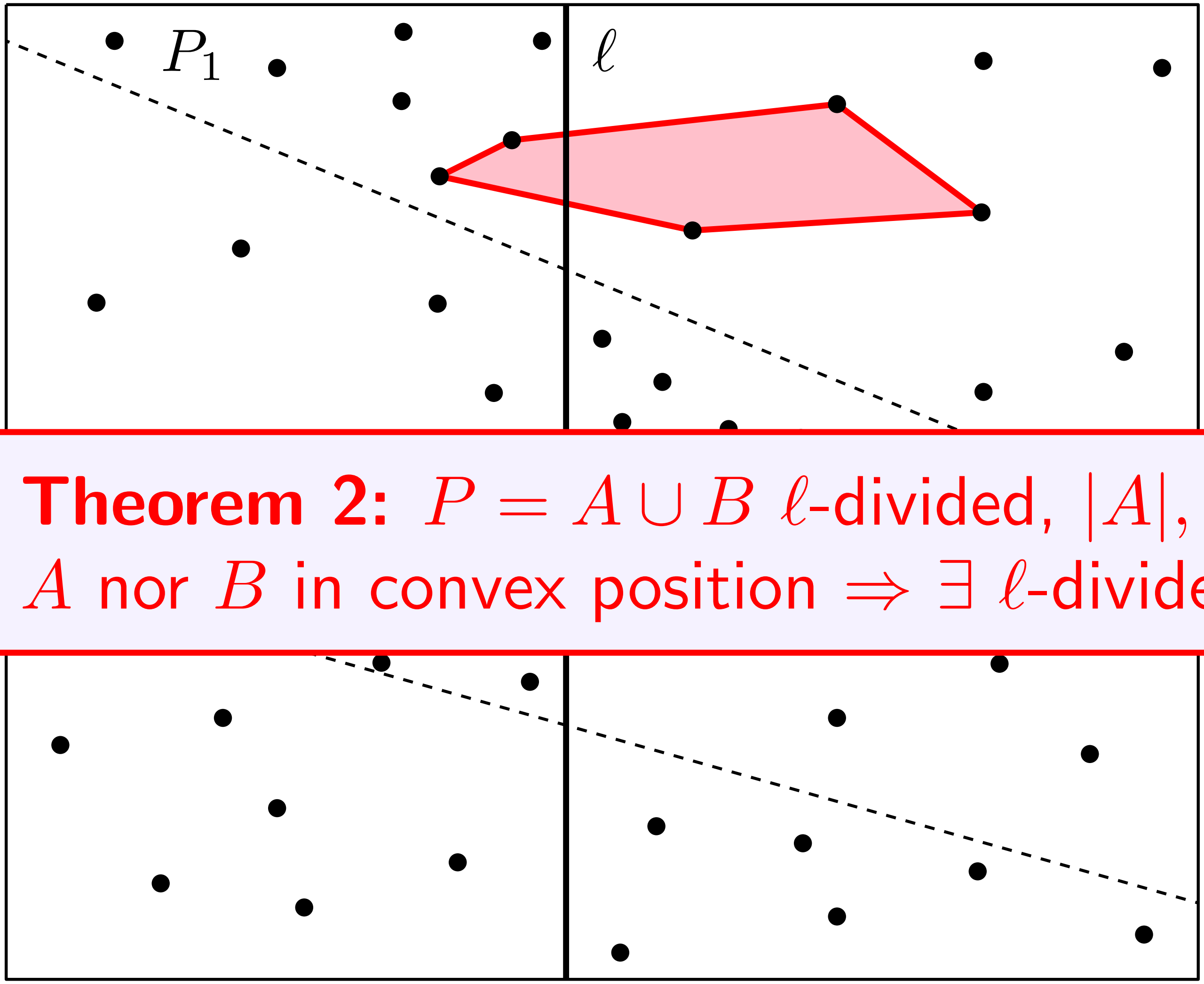

$P_i \cap A$ not convex and
$P_i \cap B$ not convex

$\overset{Th.2}{\Rightarrow} \exists\ \ell$-divided 5-hole

Theorem 2: $P = A \cup B$ ℓ-divided, $|A|, |B| \geq 5$, neither A nor B in convex position $\Rightarrow \exists\ \ell$-divided 5-hole in P

Proof of Theorem 1

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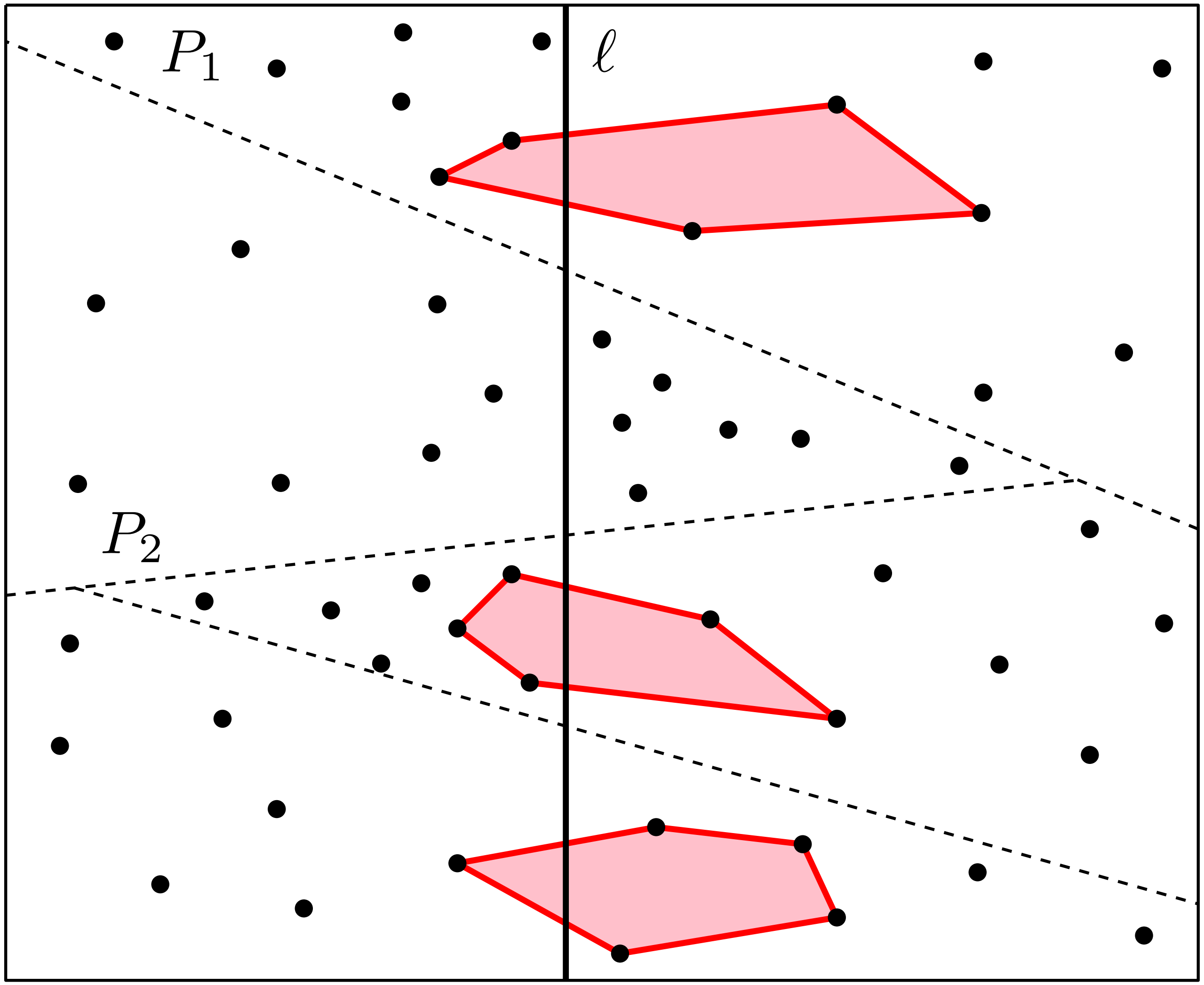


$P_i \cap A$ not convex and
$P_i \cap B$ not convex

$\overset{Th.2}{\Longrightarrow} \exists$ ℓ-divided 5-hole

If this is the case for $\frac{s}{2}$ of the islands P_i, we count at least $h_5(A) + h_5(B) + \frac{s}{2}$ 5-holes

Proof of Theorem 1

We set $r = \log^{1/5} n$. Recall that $s = \frac{n}{2r}$.

In the first case

$$h_5(P) \geq \frac{s}{2}\binom{r}{5} \geq cn \log^{4/5} n$$

and in the second case

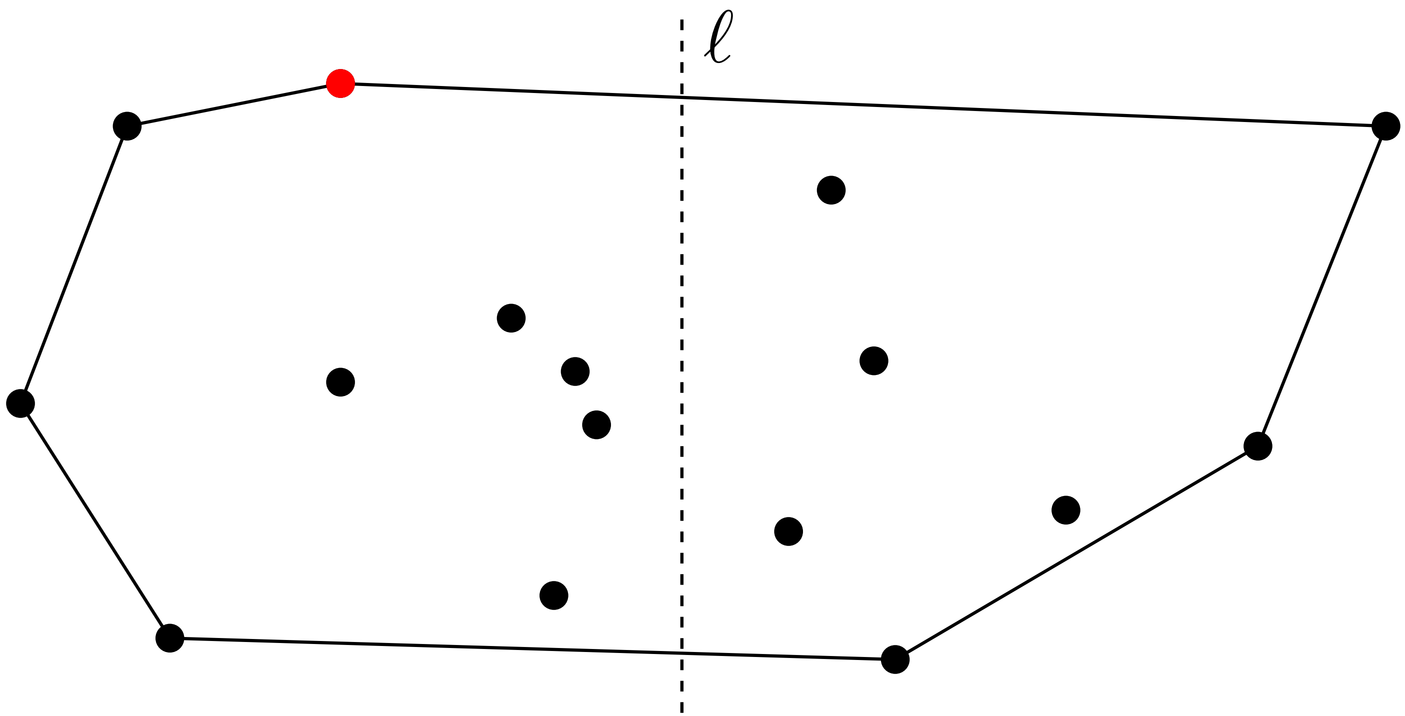
$$h_5(P) \geq h_5(A) + h_5(B) + \frac{s}{2} \overset{I.H.}{\geq} cn \log^{4/5} n.$$

This finishes the proof. $\square$

ℓ-critical sets

An ℓ-divided set $C = A \cup B$ is *ℓ-critical* if

- neither A nor B is in convex position, and
- for every extremal point x of C, one of the sets $(C \setminus \{x\}) \cap A$ and $(C \setminus \{x\}) \cap B$ is in convex position.



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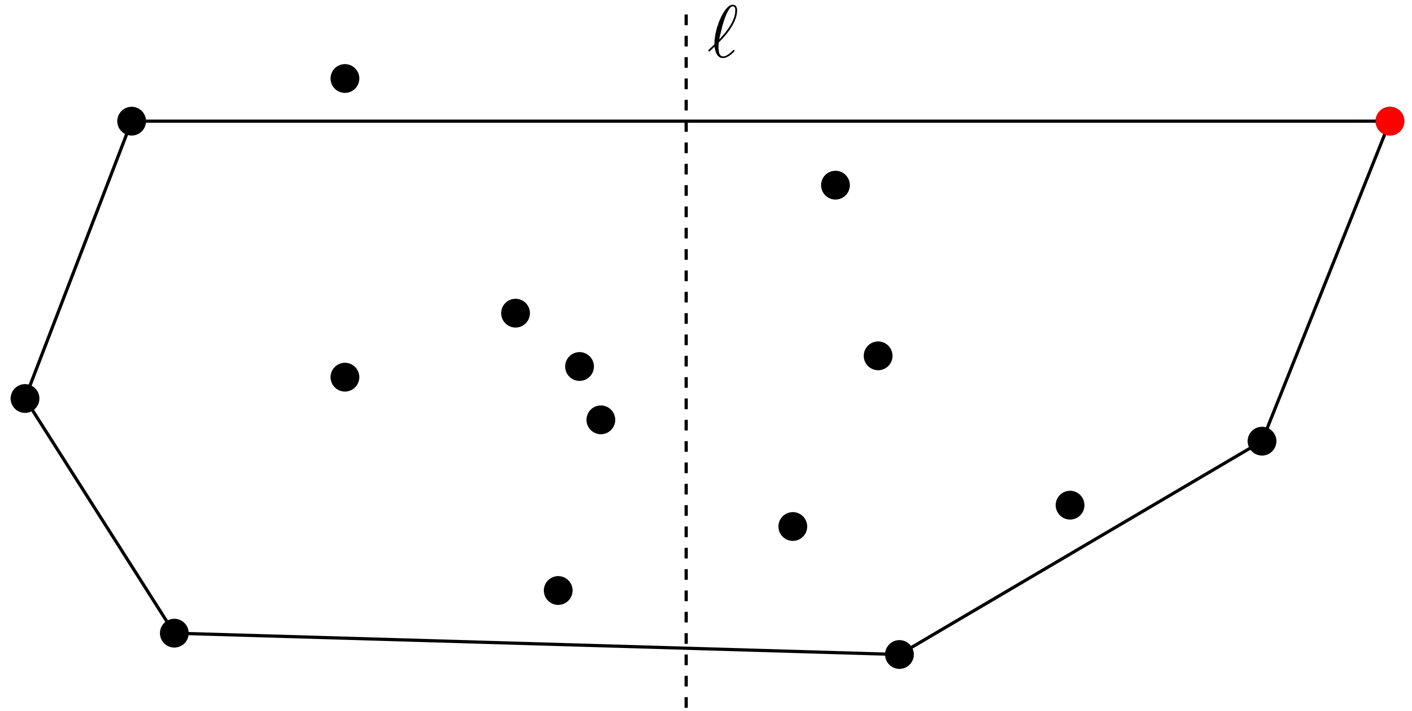

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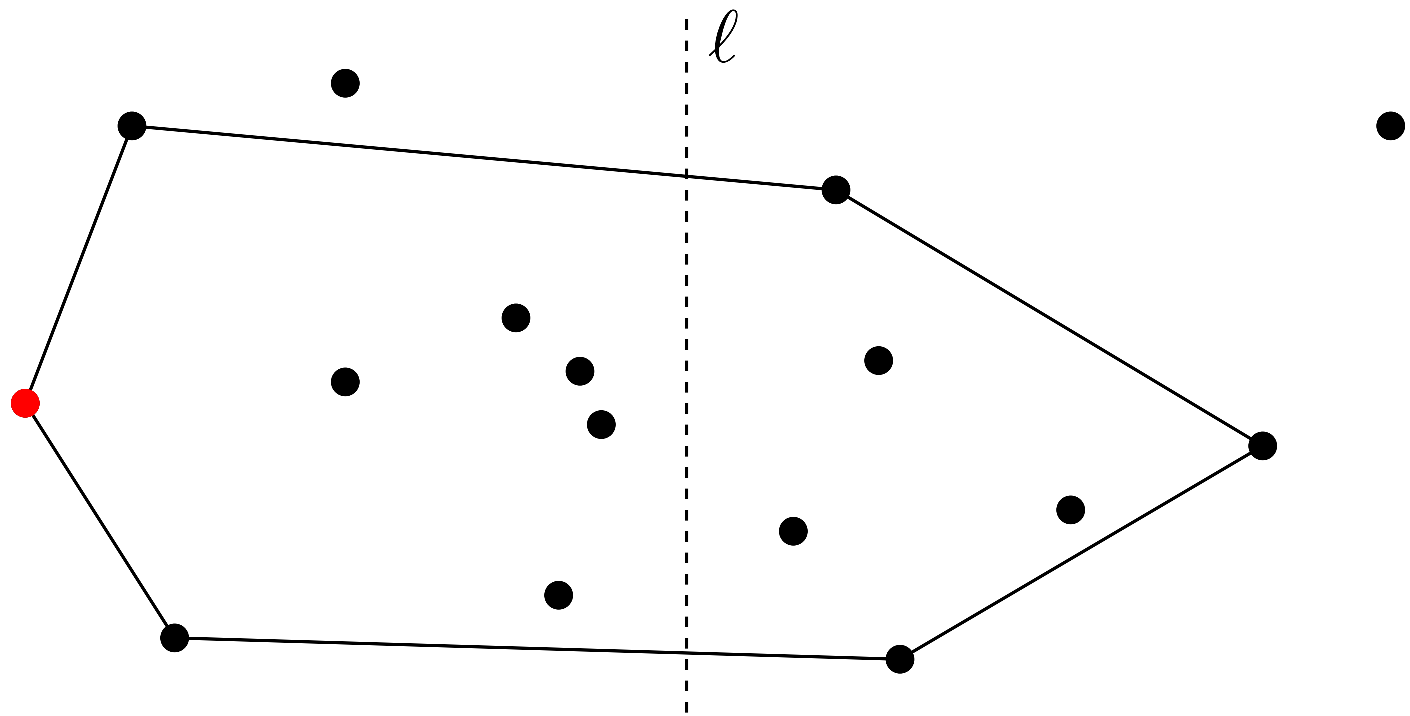

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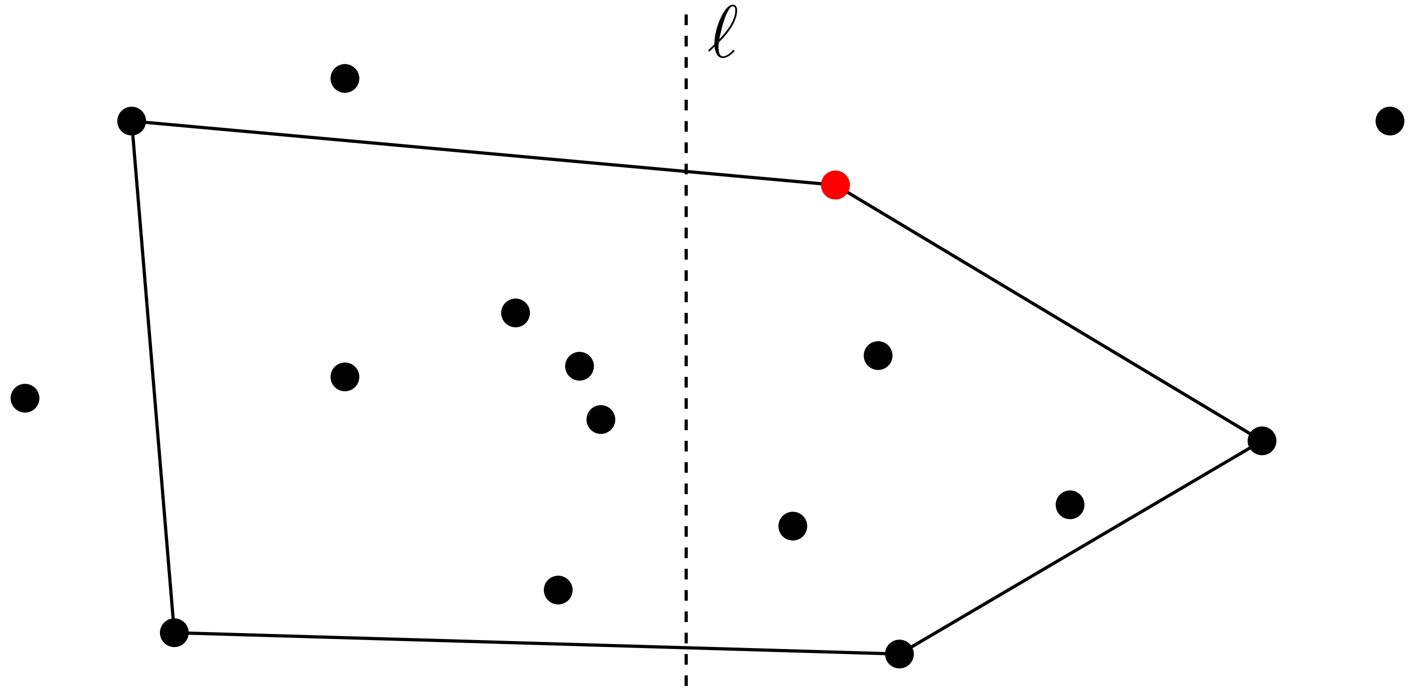

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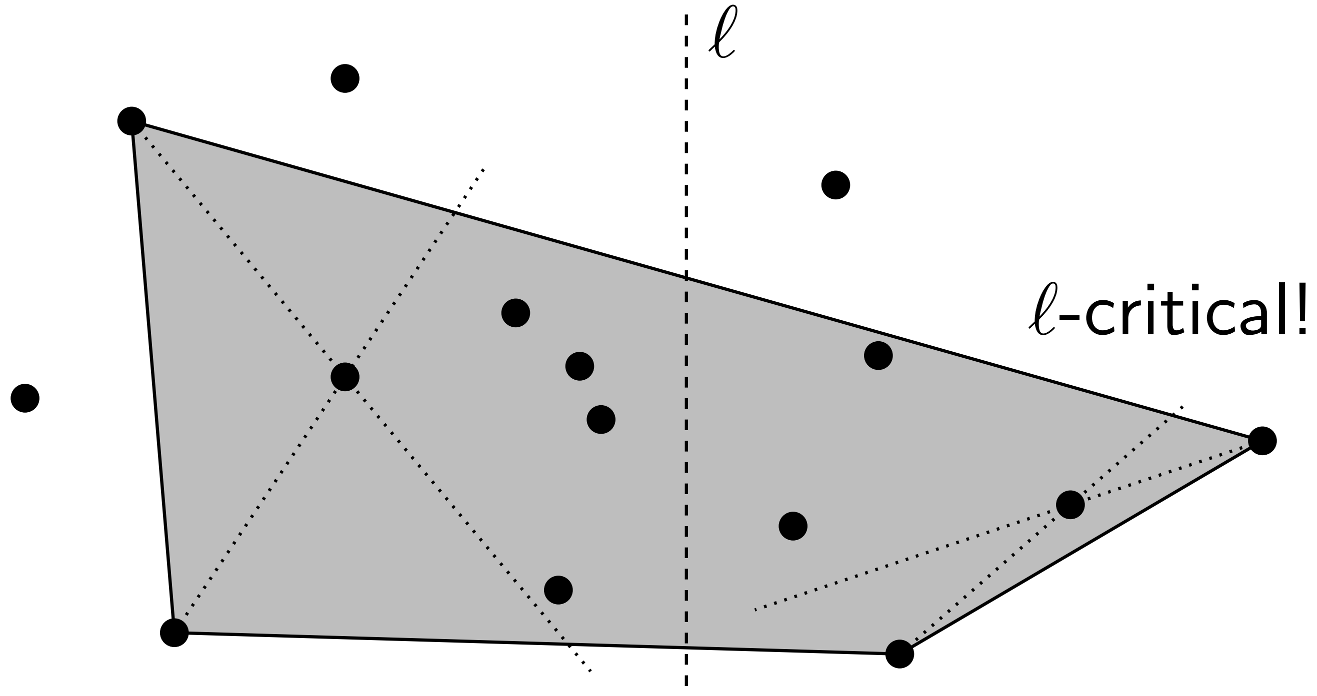

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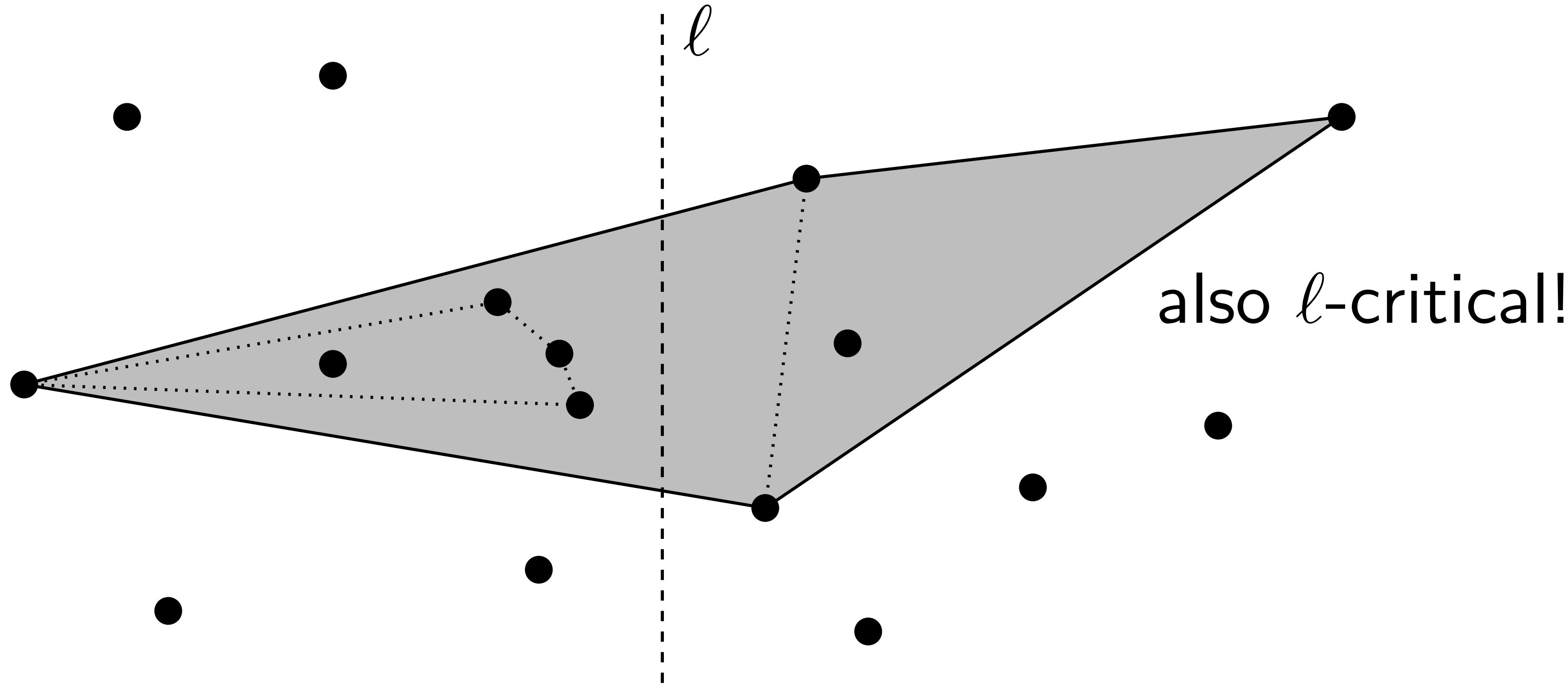
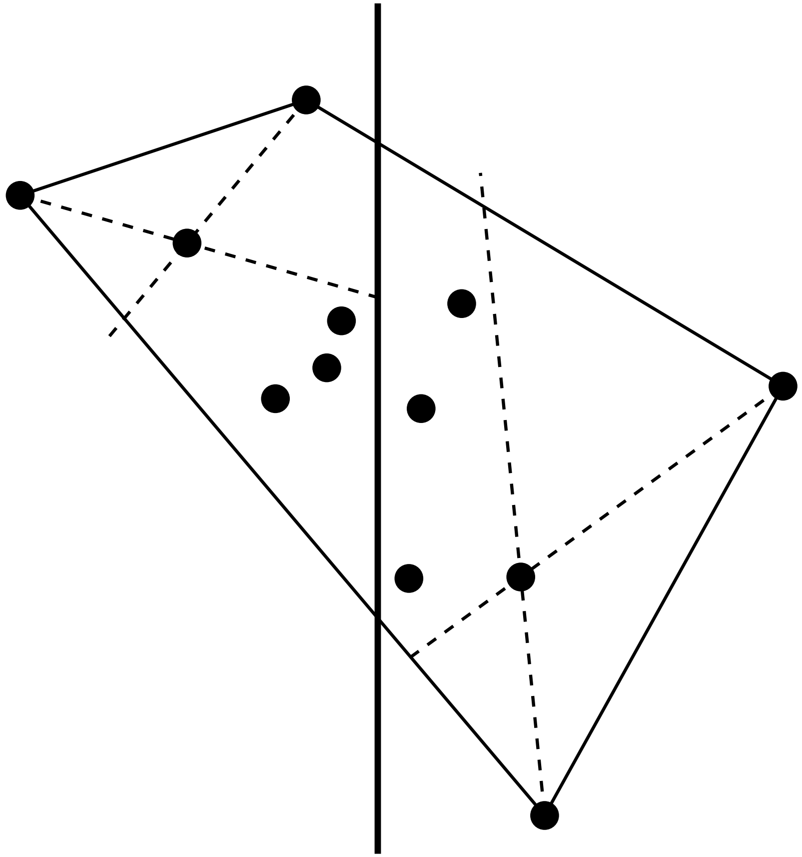


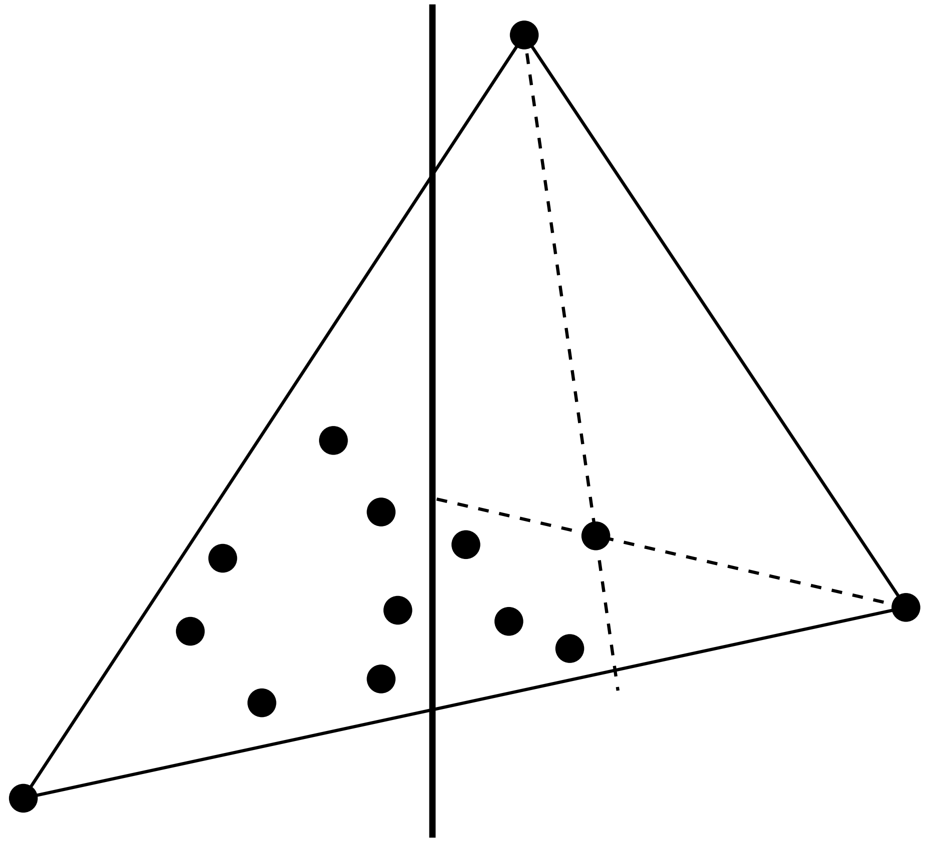
ℓ-critical sets

Lemma 18: Let $C = A \cup B$ be an ℓ-critical set.

(i) If $|A| \geq 5$, then $|A \cap \partial\,\mathrm{conv}(C)| \leq 2$ (similar for B).

(ii) If $|A \cap \partial\,\mathrm{conv}(C)| = 2$, then C looks as follows:





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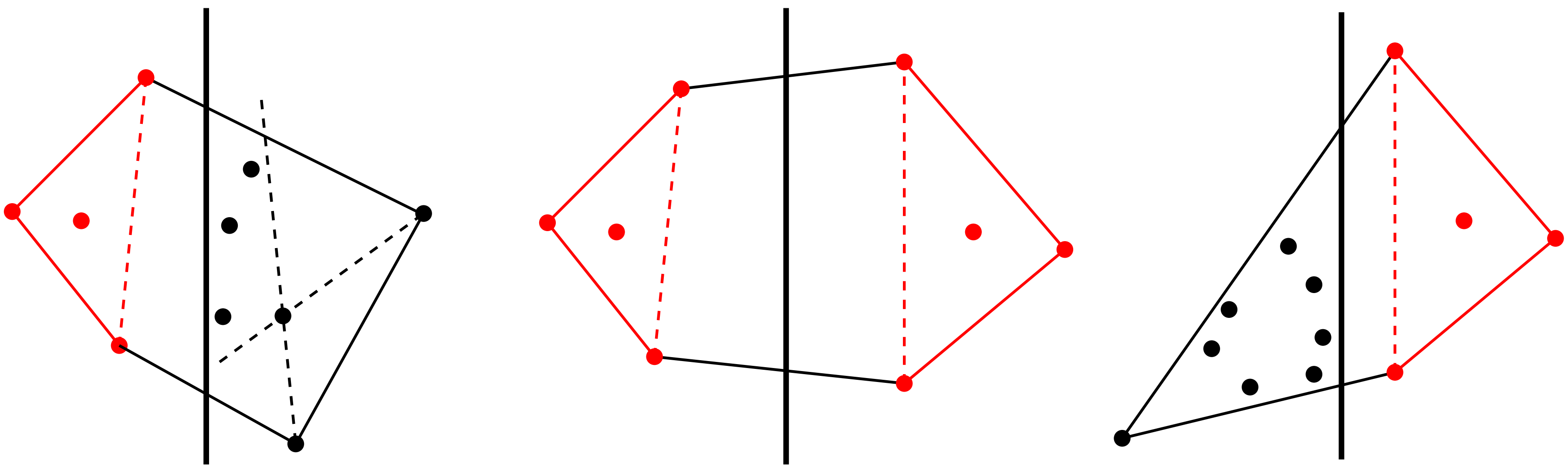
(ii) If $|A \cap \partial\,\mathrm{conv}(C)| = 2$, then C looks as follows:

The assumption $|A| \geq 5$ is necessary:



Proof of Theorem 2

Theorem 2: $P = A \cup B$ ℓ-divided, $|A|, |B| \geq 5$, neither A nor B in convex position $\Rightarrow \exists$ ℓ-divided 5-hole in P

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Proof:

- Suppose $\nexists$ ℓ-divided 5-hole.
- If $|A| = 5 = |B|$, the statement follows from Harborth's result (P contains a 5-hole).
- Hence we assume $|A| \geq 6$ or $|B| \geq 6$.

Proof of Theorem 2

We reduce P (with $|A| \geq 6$ or $|B| \geq 6$) to an island Q by iteratively removing extremal points until either

- $Q \cap A$ or $Q \cap B$ contain exactly five points, or
- Q is ℓ-critical with $|Q \cap A|, |Q \cap B| \geq 6$.

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Case 1: If $|Q \cap A| = 5$ and $|Q \cap B| \geq 6$ (or vice versa), we consider $Q \cap A$ with the 6 leftmost points of $Q \cap B$ and apply

Lemma 12 (Computer-assisted):
$P = A \cup B$ ℓ-divided, $|A| = 5$, $|B| = 6$, A not in convex position $\Rightarrow \exists$ ℓ-divided 5-hole.

Proof of Theorem 2

Case 2: If Q is ℓ-critical with $|Q \cap A|, |Q \cap B| \geq 6$, then, by Lemma 18, $|A \cap \partial\,\mathrm{conv}(Q)| = 2$ (w.l.o.g.)

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Proposition 19+20: $C = A \cup B$ ℓ-critical, no ℓ-divided 5-hole, $|A|, |B| \geq 6$

(i) $|A \cap \partial \operatorname{conv}(C)| = 2 \Rightarrow |B| \leq |A| - 1.$

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> **Proposition 19+20:** $C = A \cup B$ ℓ-critical, no ℓ-divided 5-hole, $|A|, |B| \geq 6$
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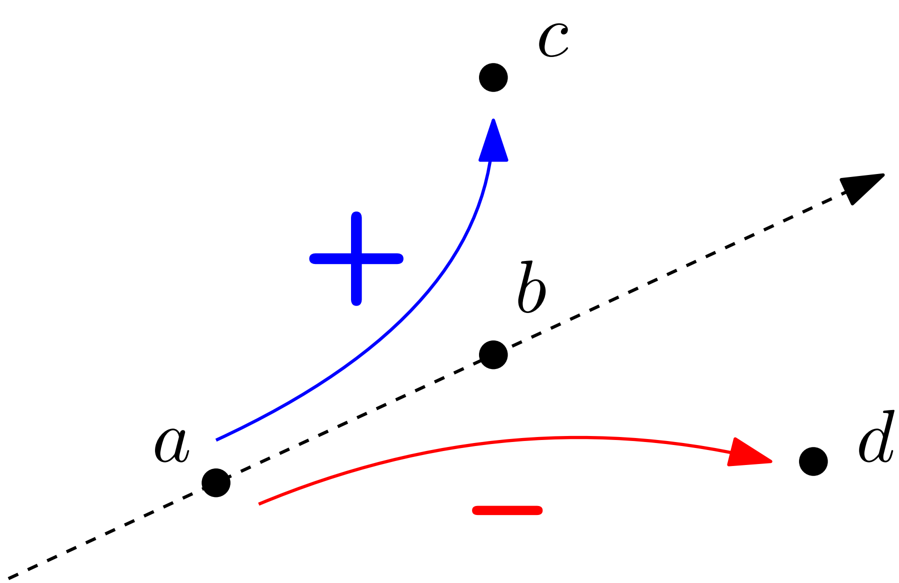
By exchanging the roles of A and B, we obtain $|Q \cap A| \leq |Q \cap B|$ – a contradiction $\square$

Order Types of Point Sets

Three distinct points a, b, c in P either *positively oriented* or *negatively oriented*



$$\sigma_{abc} = +$$
$$\sigma_{abd} = -$$

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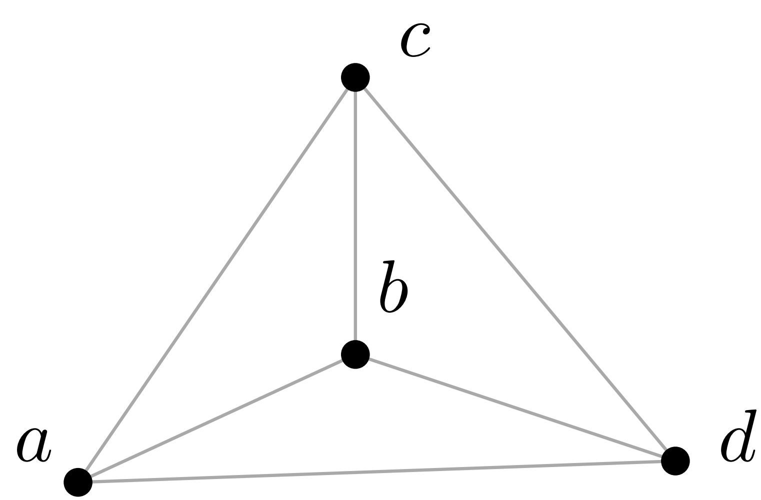
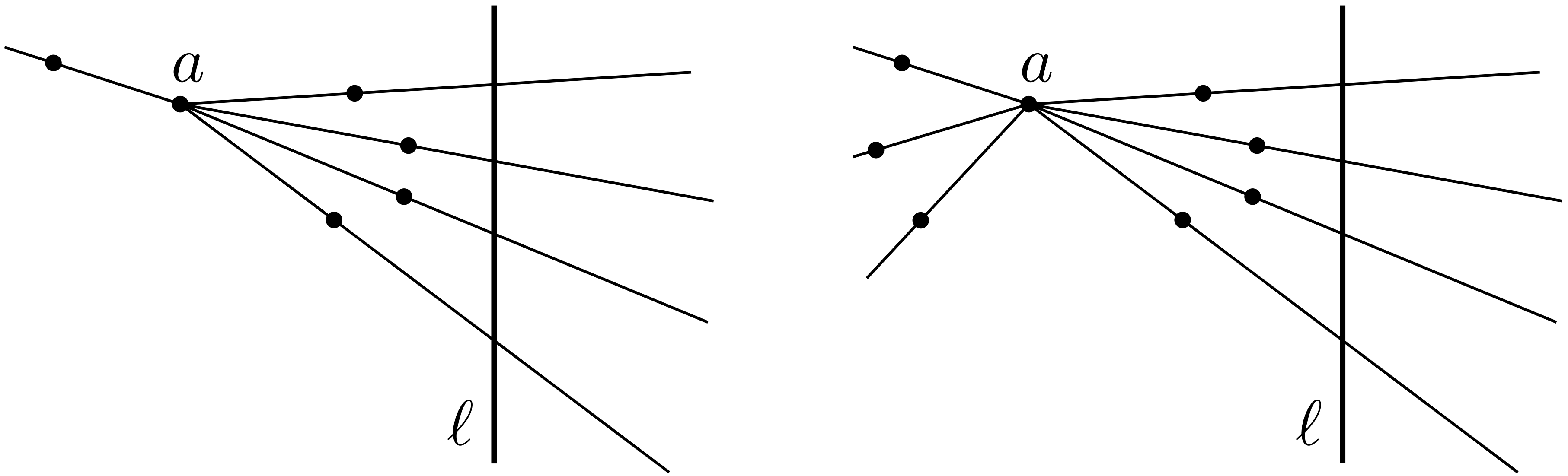


$$\sigma_{abc} = +$$
$$\sigma_{abd} = -$$
$$\sigma_{acd} = -$$
$$\sigma_{bcd} = -$$

Order types: distinguish point sets only by orientations of triples [Goodman and Pollack '83]

a-wedges

- For a in A, the rays $\overrightarrow{aa'}$ $(a' \in A \setminus \{a\})$ partition the plane into $|A| - 1$ regions
- *a-wedges*: closures of those regions

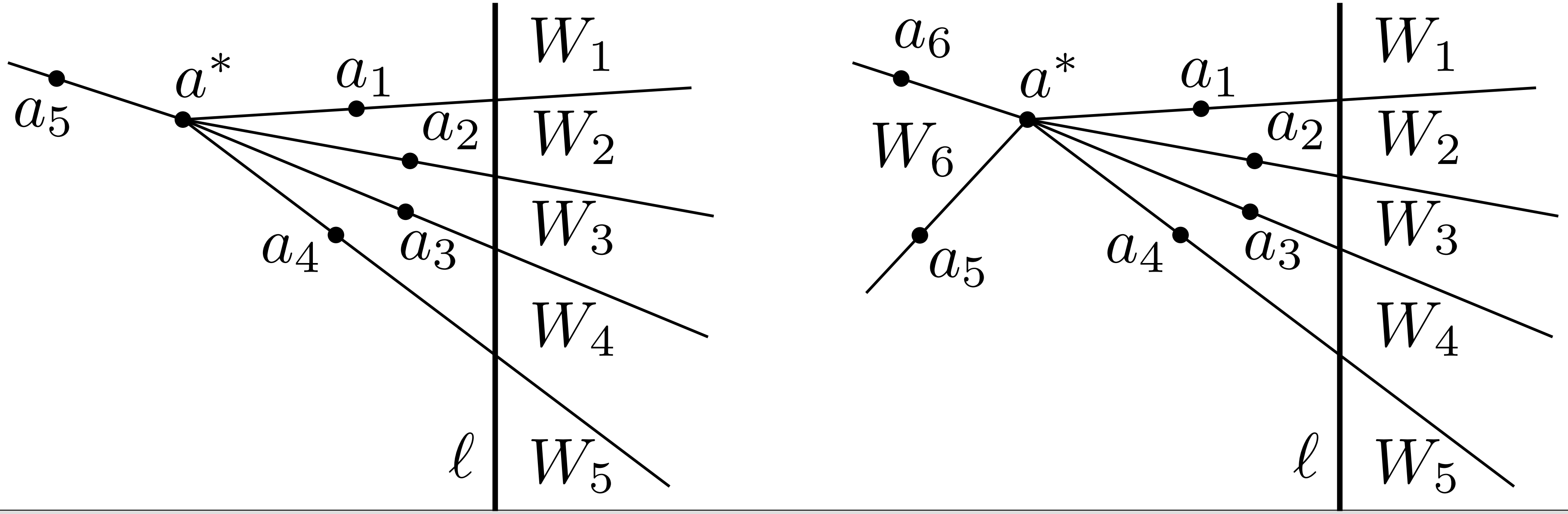


- a-wedges are convex if a is inner point of A and $\exists!$ non-convex a-wedge otherwise.

a^*-wedges

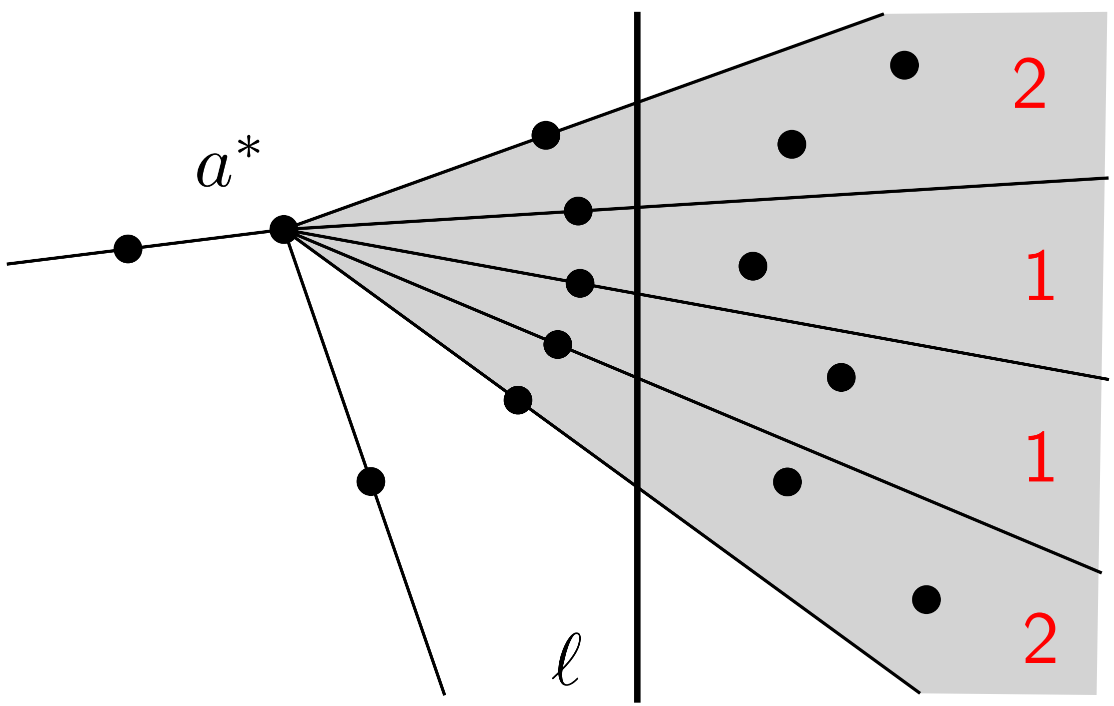
If A not in convex position

- a^*: rightmost inner point of A,
- $W_1, \ldots, W_{|A|-1}$: a^*-wedges
- $w_i = |B \cap W_i|$
- $t = \#$ of a^*-wedges intersecting ℓ,
- $a_1, \ldots, a_{|A|-1}$: points in $A \setminus \{a^*\}$.



Sequences of a^*-wedges

Lemma 10: $P = A \cup B$ ℓ-divided, no ℓ-divided 5-hole, A not in convex position, $|A| \geq 5$, $|B| \geq 6$, $(w_i, \ldots, w_j) = (2, 1, \ldots, 1, 2) \Rightarrow \exists$ ℓ-divided 5-hole.



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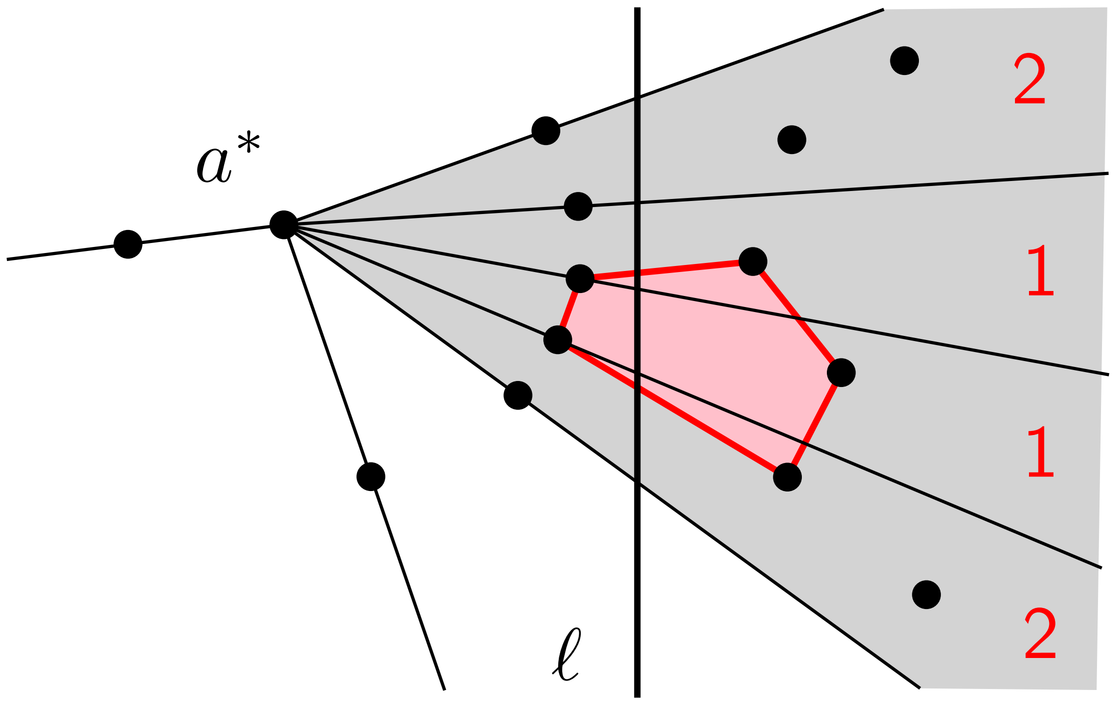
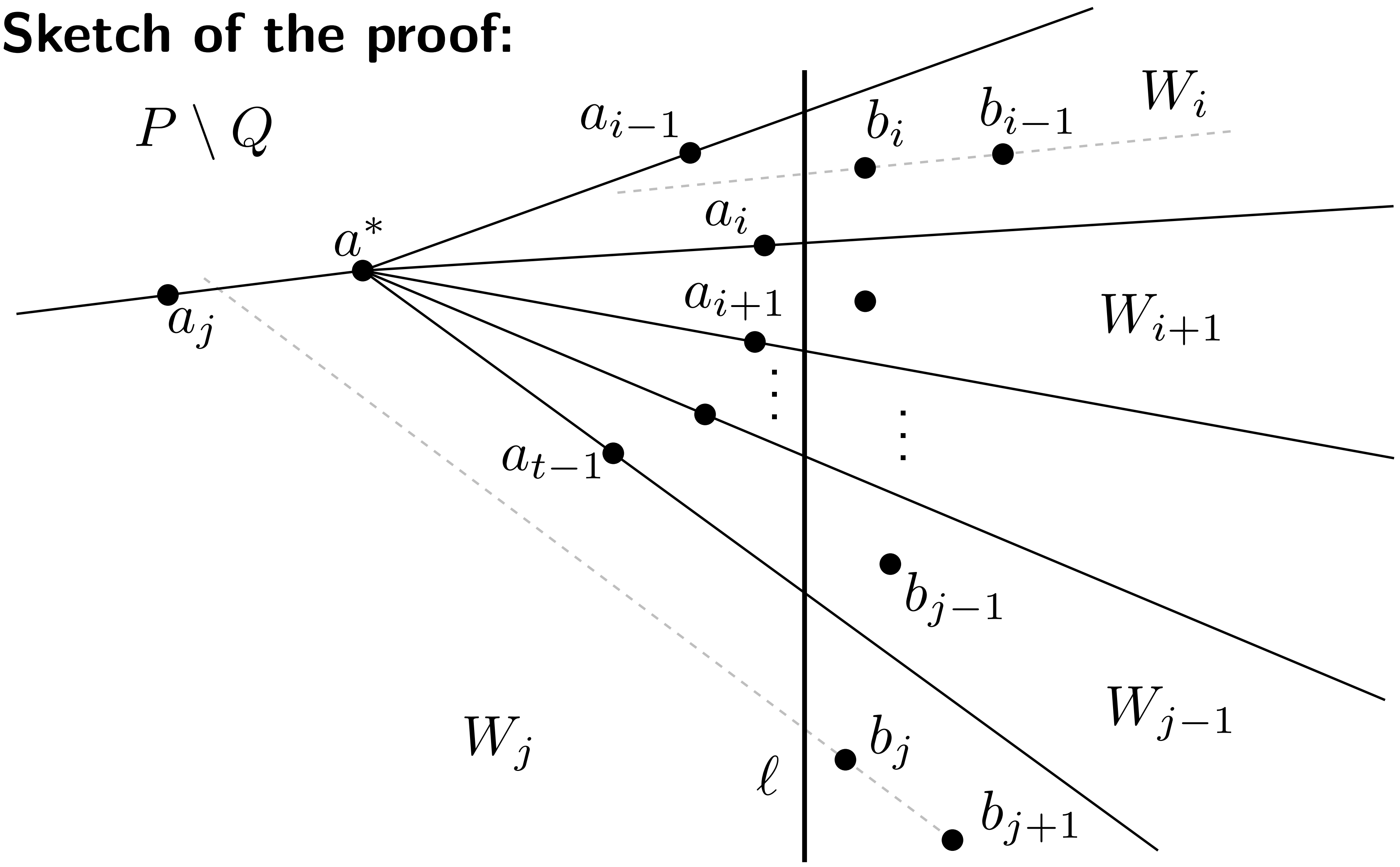


Sequences of a^*-wedges

Sketch of the proof:



Sequences of a^*-wedges

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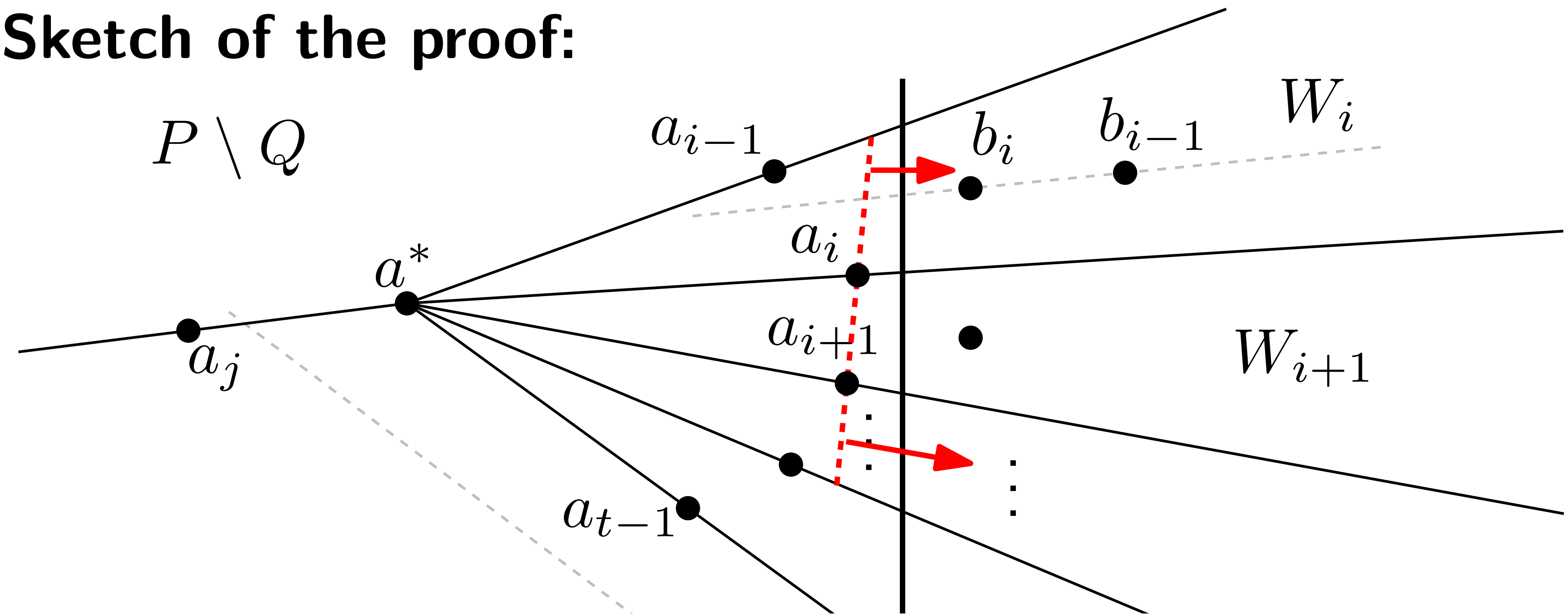

- $B \cap (W_{i-1} \cup W_i \cup W_{i+1})$ to the right of $\overline{a_i a_{i-1}}$

Sequences of a^*-wedges

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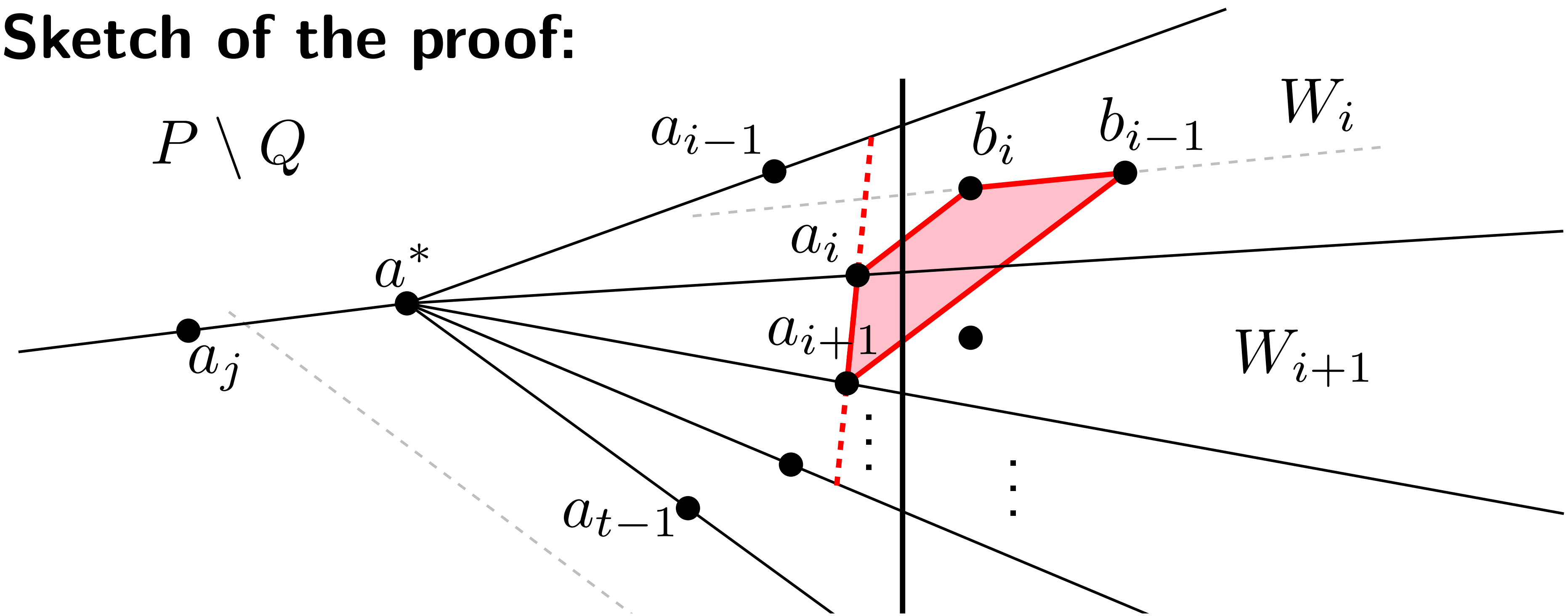

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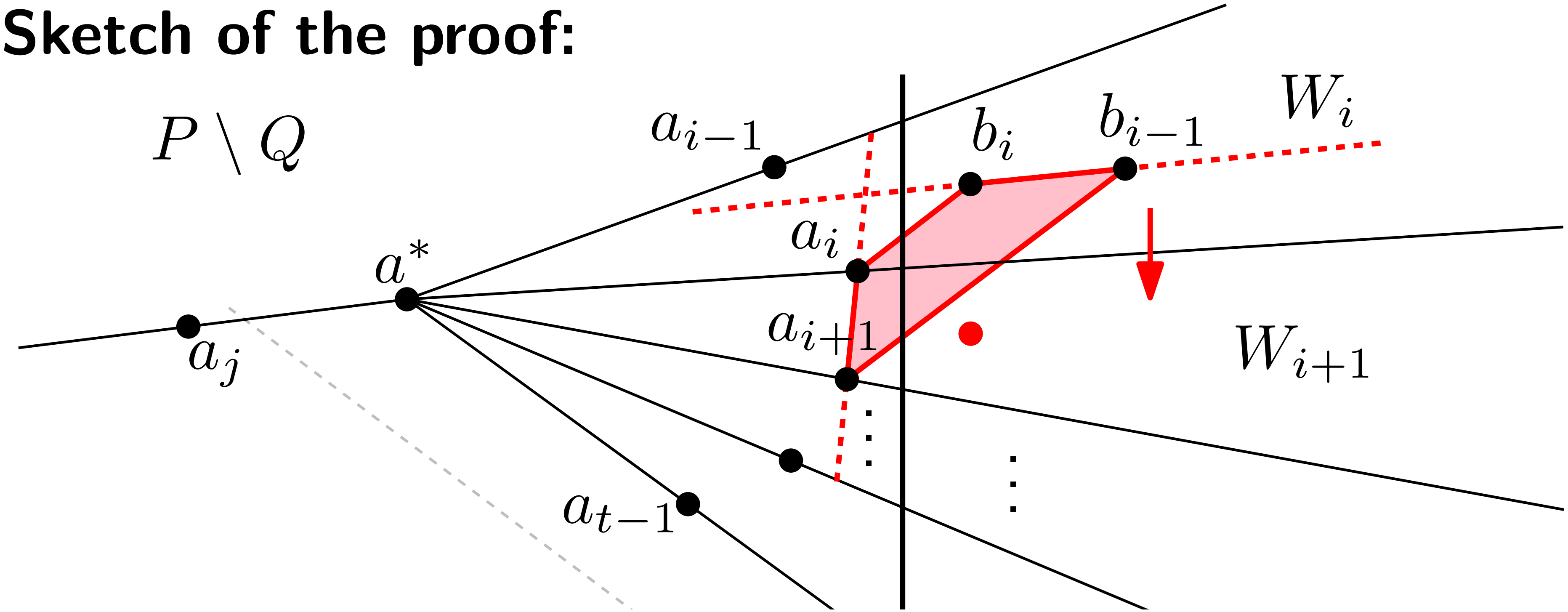
- $b_{i-1}, b_i, a_i, a_{i+1}$ form convex quadrilateral

Sequences of a^*-wedges

Sketch of the proof:



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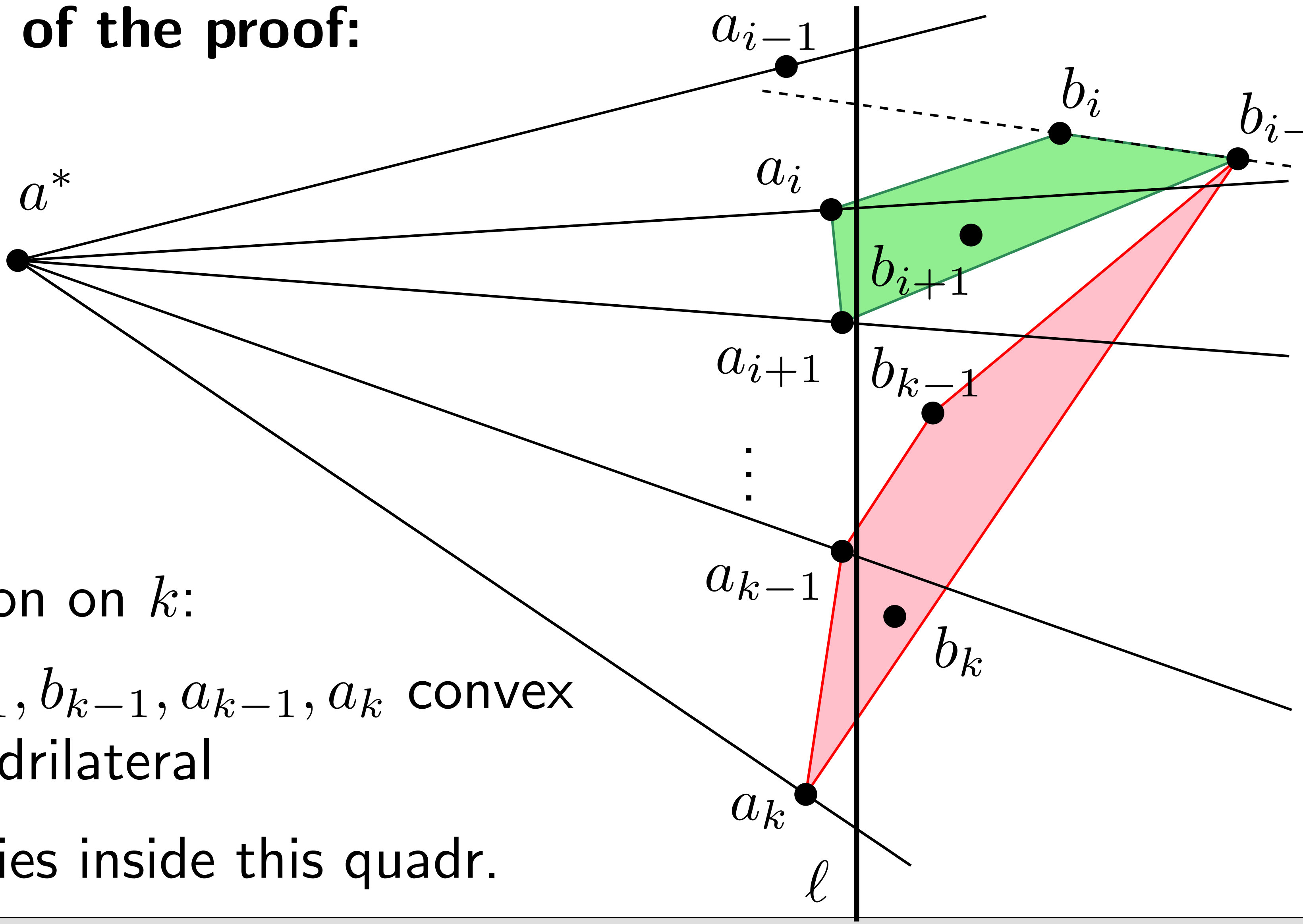
- b_{i+1} to the right of $\overline{b_i b_{i-1}}$

Sequences of a^*-wedges

Sketch of the proof:

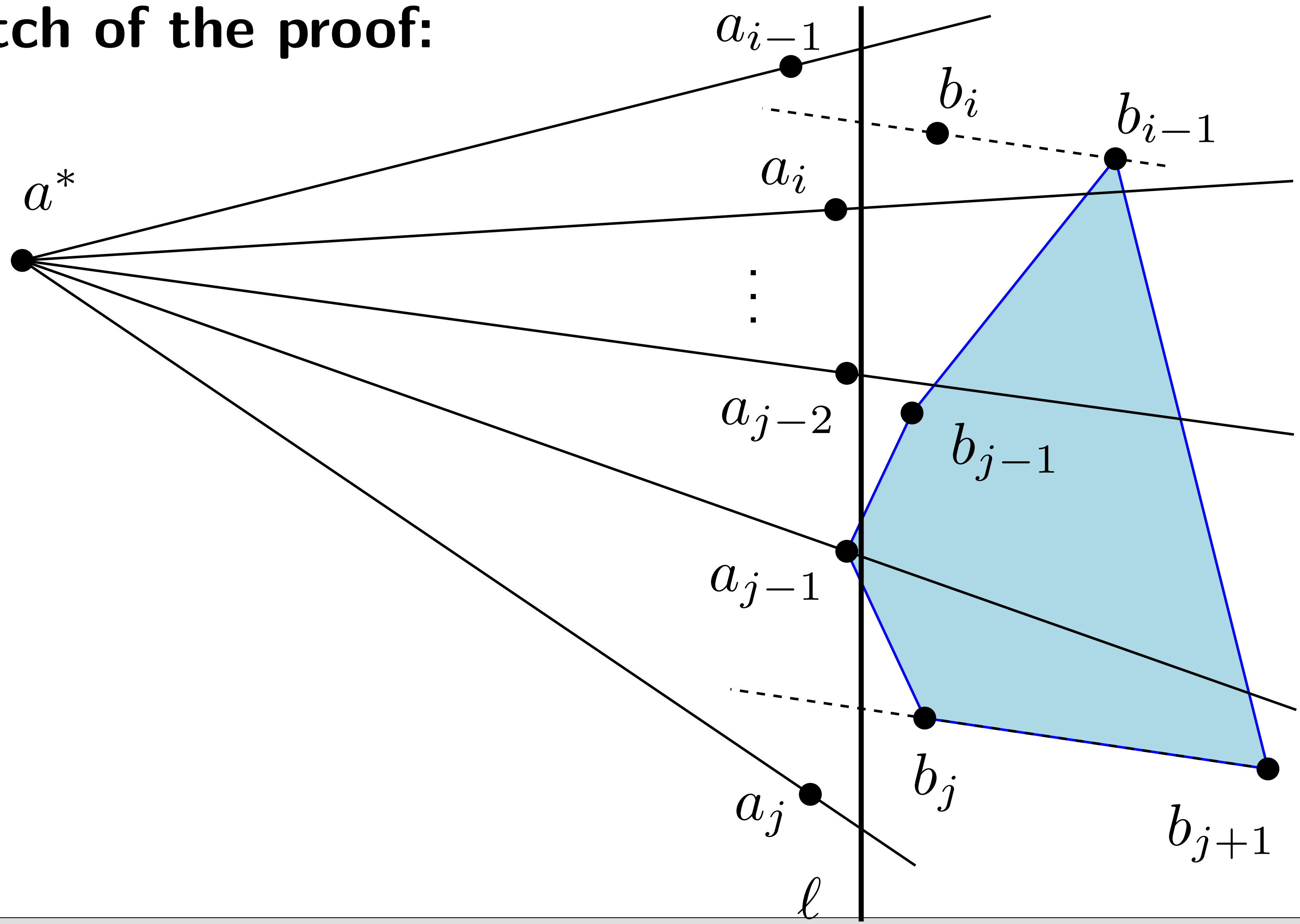


Induction on k:

- $b_{i-1}, b_{k-1}, a_{k-1}, a_k$ convex quadrilateral
- b_k lies inside this quadr.

Sequences of a^*-wedges

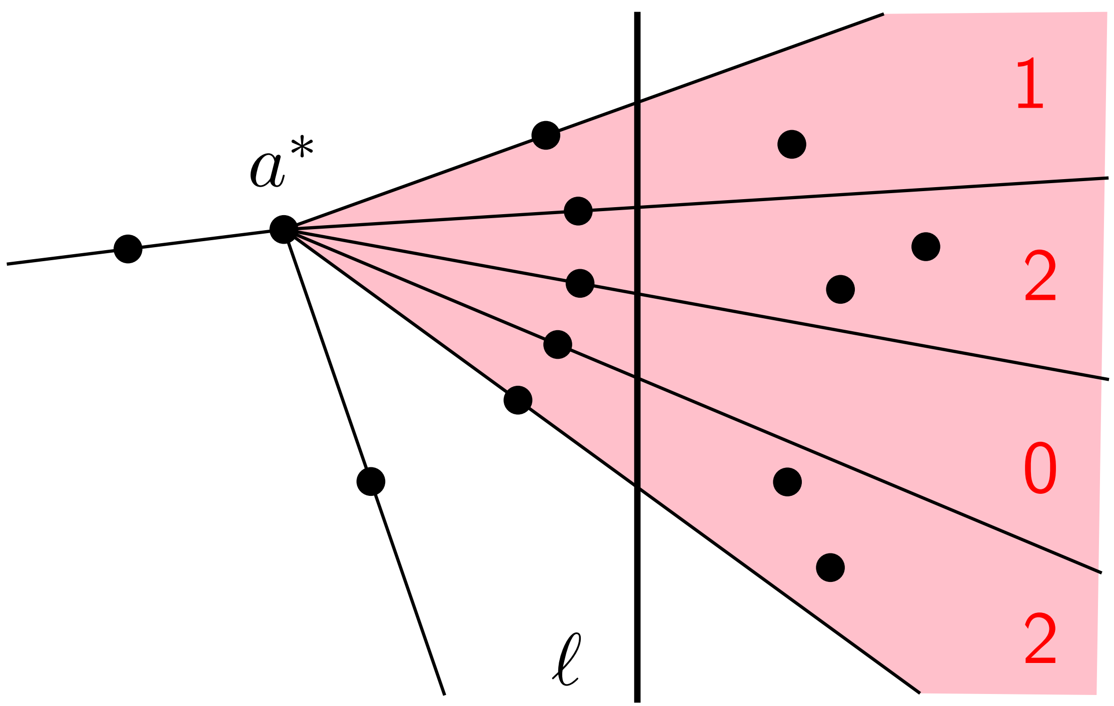
Sketch of the proof:



$\square$

Sequences of a^*-wedges

Proposition 11: $P = A \cup B$ ℓ-divided, no ℓ-divided 5-hole, A not in convex position, $|A| \geq 5$, $|B| \geq 6$, $w_k \leq 2$ for $i \leq k \leq j \Rightarrow \sum_{k=i}^{j} w_k \leq j - i + 2$.

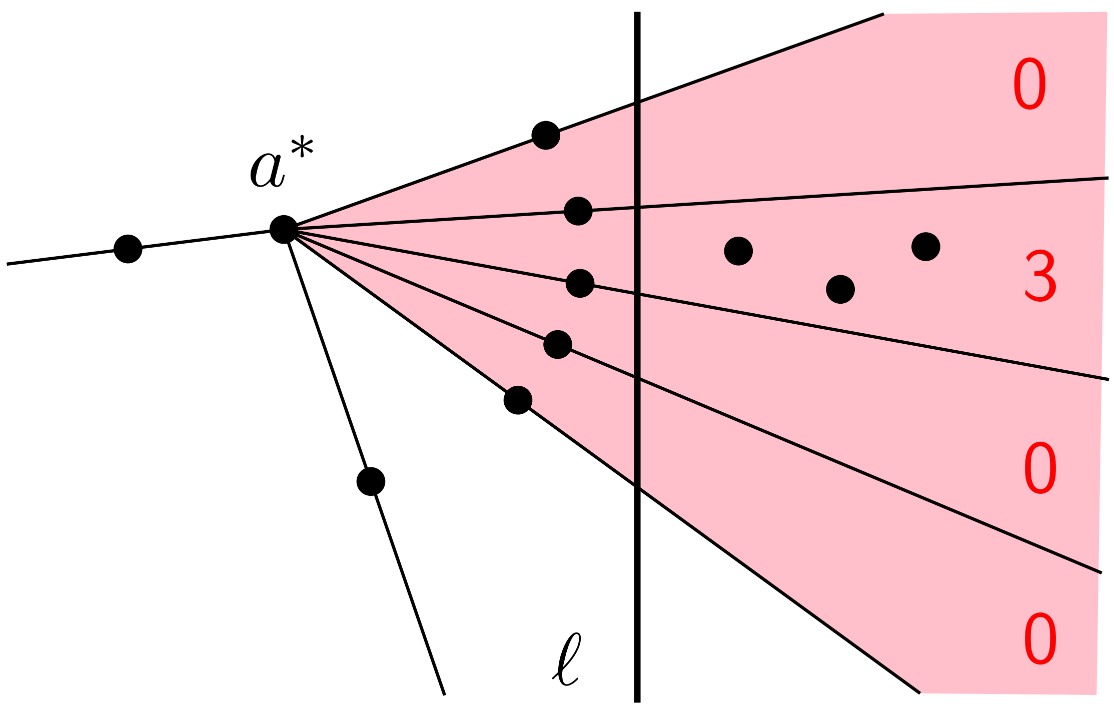


Sequences of a^*-wedges

Lemma 16: $P = A \cup B$ ℓ-divided, no ℓ-divided 5-hole, $|A| \geq 6$, A not in convex position

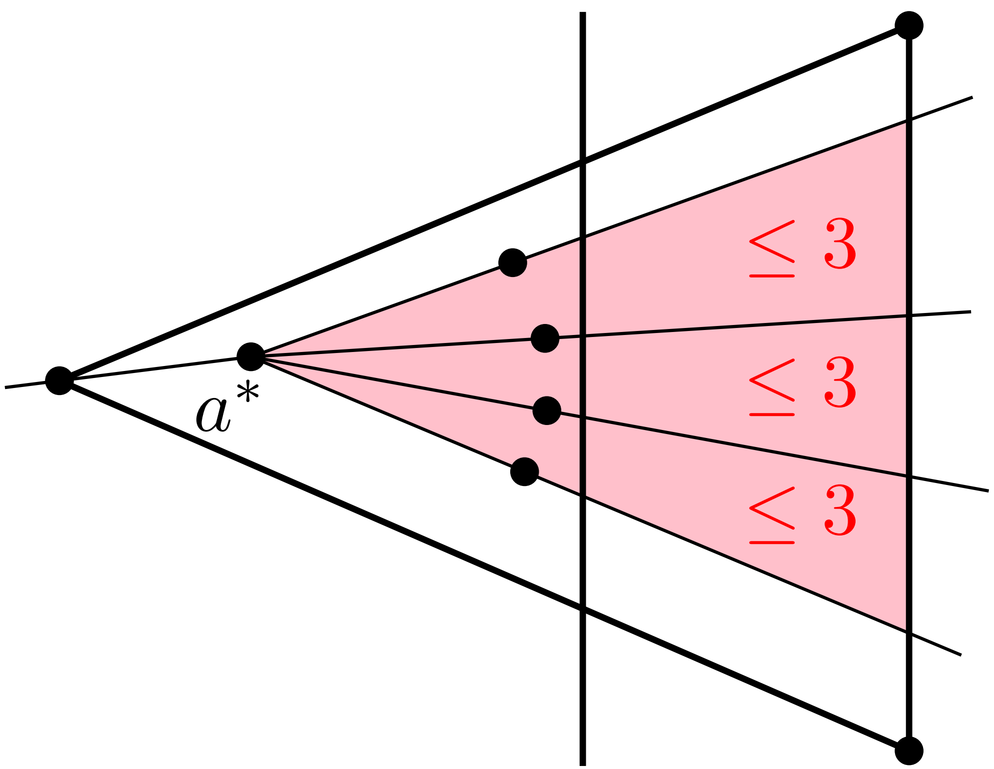
(i) $w_i + w_{i+1} + w_{i+2} \geq 4 \Rightarrow w_i, w_{i+1}, w_{i+2} \leq 2$.

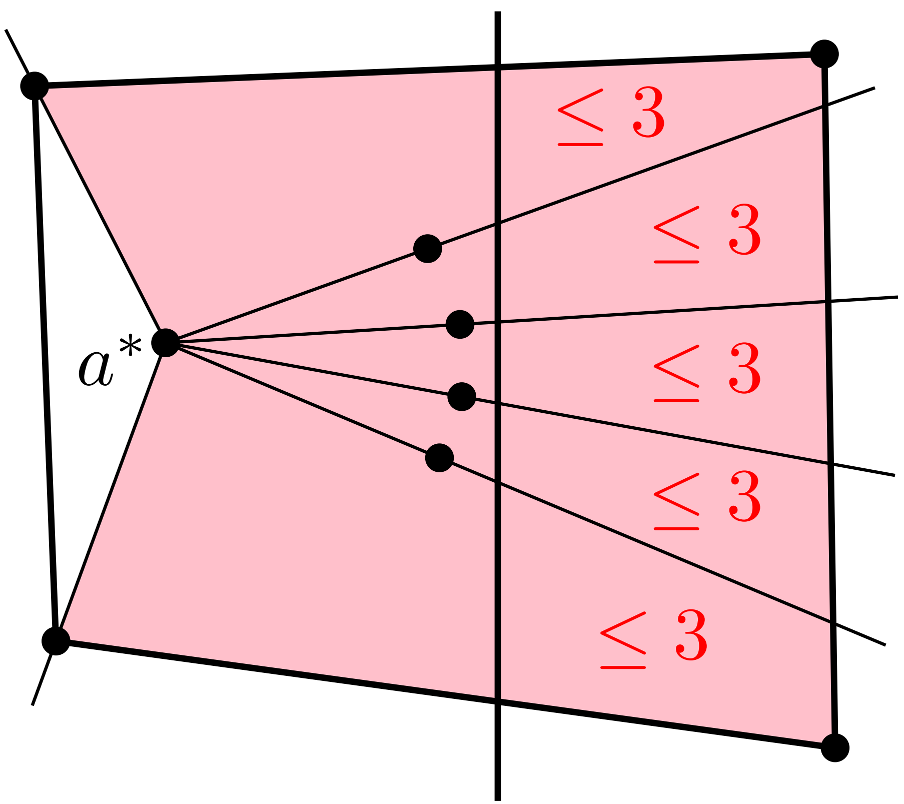
(ii) $w_i + \ldots + w_{i+3} \geq 4 \Rightarrow w_i, \ldots, w_{i+3} \leq 2$.



Sequences of a^*-wedges

Lemma 17: $C = A \cup B$ ℓ-critical, no ℓ-divided 5-hole, $|A| \geq 6 \Rightarrow w_i \leq 3$ for every $1 < i < t$.
Moreover, $|A \cap \partial\,\mathrm{conv}(C)| = 2 \Rightarrow w_1, w_t \leq 3$.





Sequences of a^*-wedges

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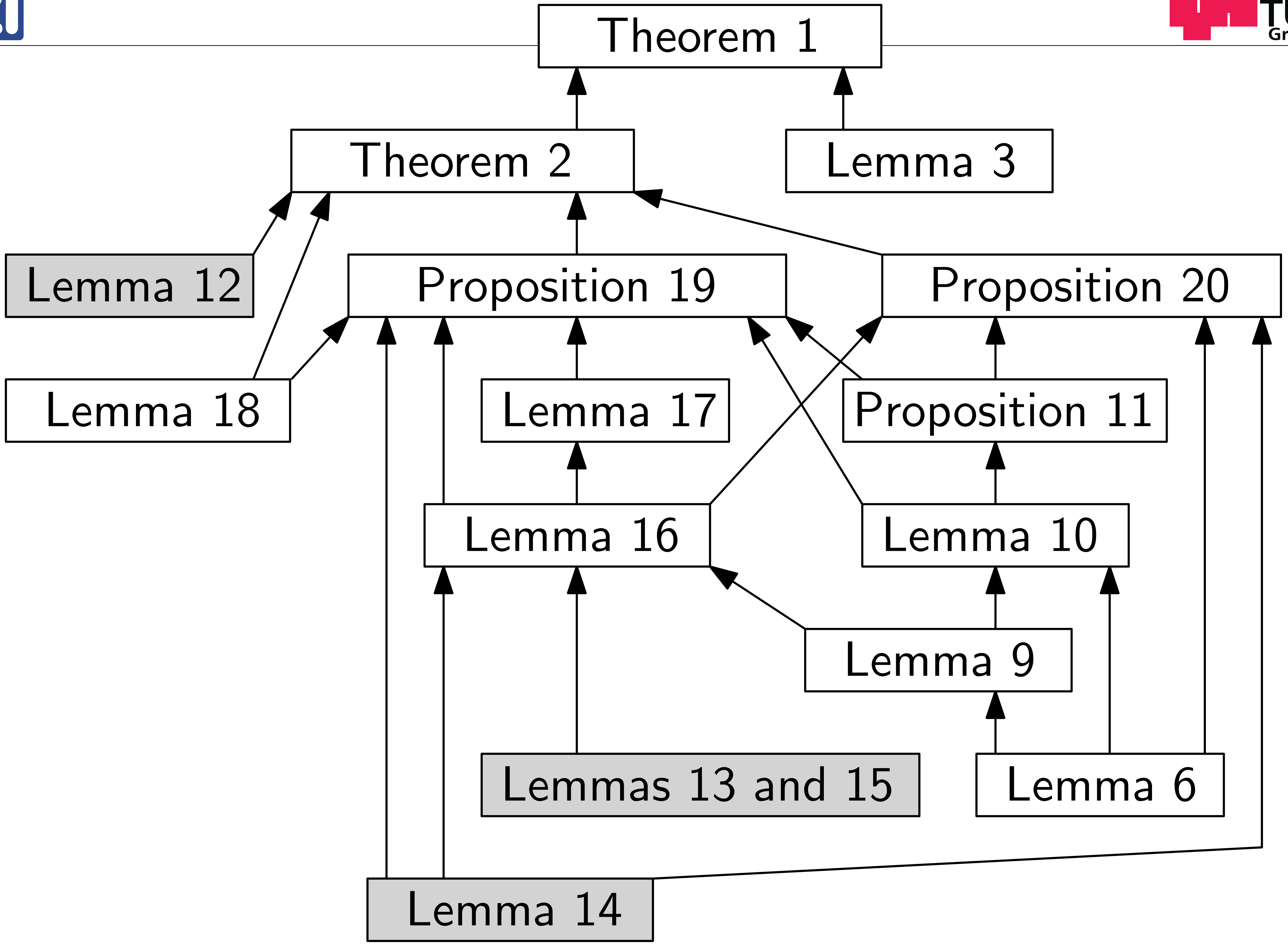
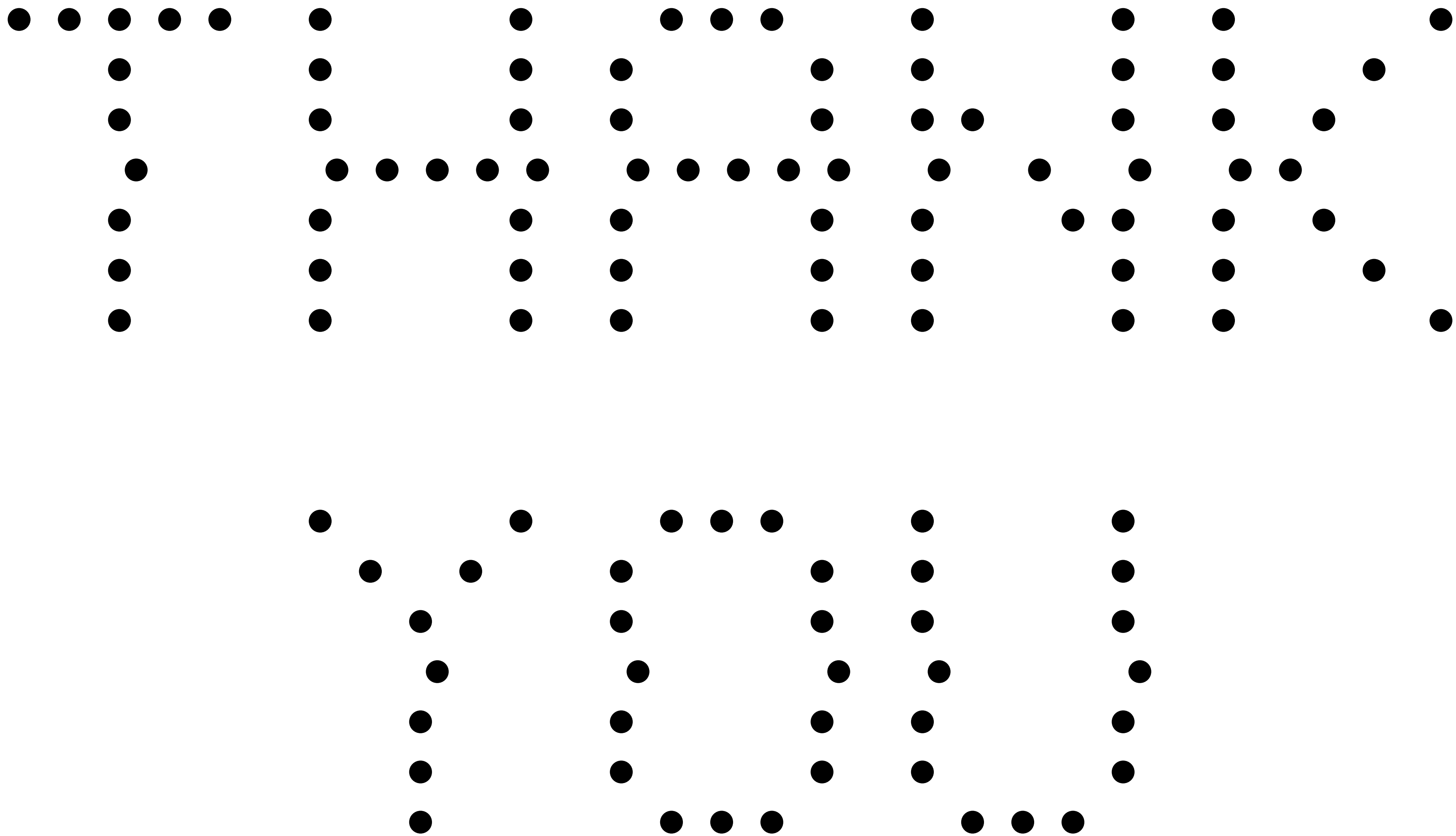
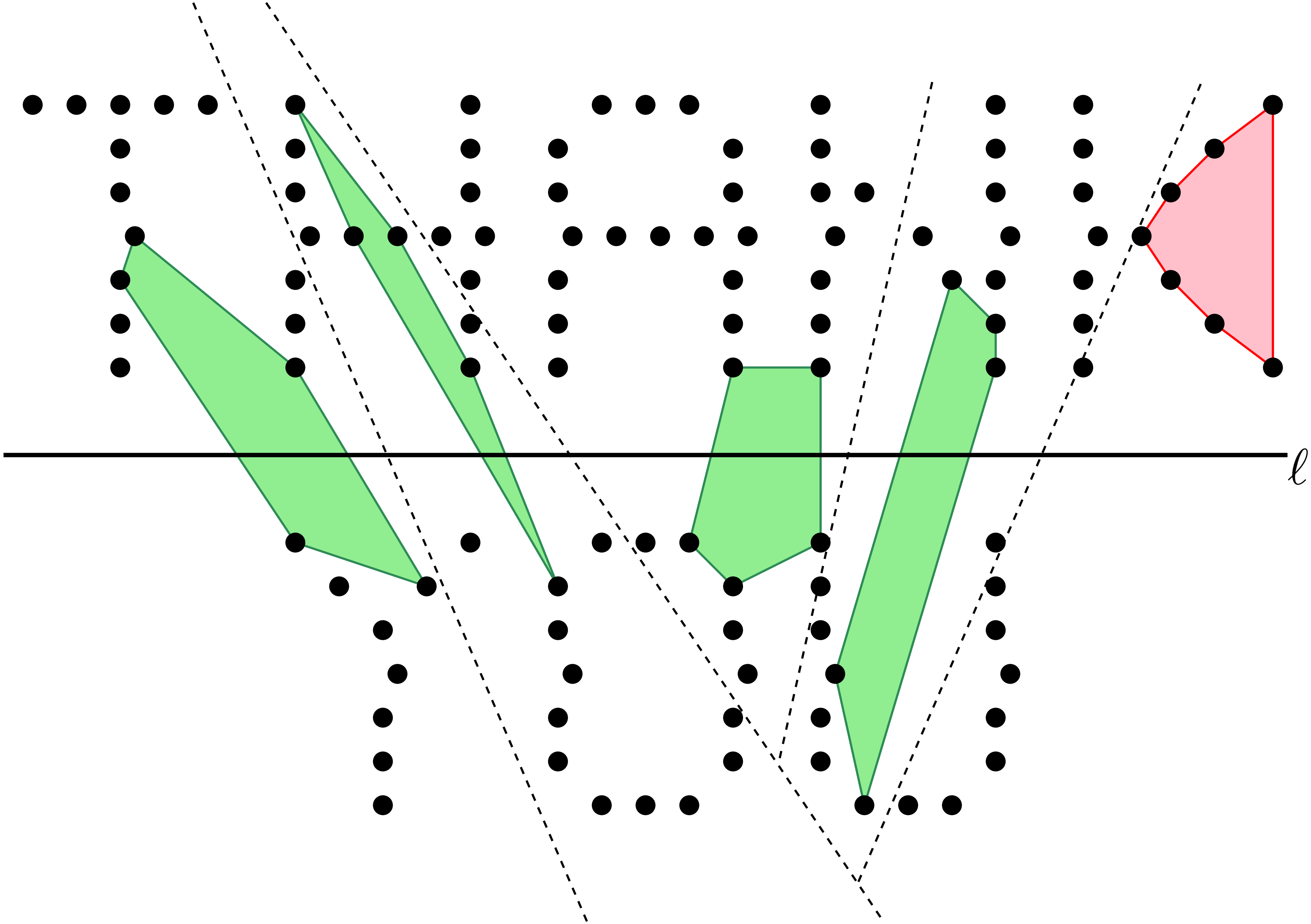
Proposition 19: $C = A \cup B$ ℓ-critical, no ℓ-divided 5-hole, $|A|, |B| \geq 6$, $|A \cap \partial\,\mathrm{conv}(C)| = 2 \Rightarrow |B| \leq |A| - 1$.

Proposition 20: $C = A \cup B$ ℓ-critical, no ℓ-divided 5-hole, $|A|, |B| \geq 6$, $|B \cap \partial\,\mathrm{conv}(C)| = 2 \Rightarrow |B| \leq |A|$.


Theorem 1
Theorem 2
Lemma 3
Lemma 12
Proposition 19
Proposition 20
Lemma 18
Lemma 17
Proposition 11
Lemma 16
Lemma 10
Lemma 9
Lemmas 13 and 15
Lemma 6
Lemma 14


THANK
YOU



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