The price of spite in Spot Checking Games

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joint work wit Ralf Borndörfer, Thomas Schlechte, Elmar Swarat.

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Sep 30, University of Patras
Truck toll in Germany

A Huge and original system

- Started in 2005
- Tax based on distance and truck category (avg price: 0.17 Euro/km)
- Total revenue around 3 Billion Euros / year
- No toll barriers
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Goal of this work
Study optimal patrolling strategy to enforce the payment of tolls in transportation networks, from a game-theoretic point of view
1 Spot checking games

2 Computation of Equilibria
   - Nash Equilibria: LP
   - Computing a Stackelberg Equilibrium is NP-hard

3 The price of Spite

4 Numerical Results
A simple, general model

Notation

\[ G = (V, E) \quad : \quad \text{Graph of the network} \]
\[ C \quad : \quad \text{set of weighted commodities } (s_k, t_k, x_k)_{k \in K} \]
\[ w_e \quad : \quad \text{cost for taking } e \in E \text{ (travel charge + toll fare)} \]
A simple, general model

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\[ w_e \] : cost for taking \(e \in E\) (travel charge + toll fare)

\[ q_e \] : Probability that an inspector is on section \(e \in E\)
\[ \sigma_e = \pi_e P \] : Expected penalty to pay for evading on \(e\), conditionally to the presence of an inspector

\[ \beta_e \] : Reward for the inspector for each user taking \(e\)
1. Transit system with fixed access fare $T$. 
Model examples: underlying Graph

1. Transit system with fixed access fare \( T \).

\[
\begin{align*}
    w_1 &= c_1 \\
    \sigma_1 q_1 &= 0 \\
    w_2 &= c_2 \\
    \sigma_2 q_2 &= 0 \\
    w_3 &= c_3 \\
    \sigma_3 q_3 &= 0 \\
    w_{ad} &= c_1 + c_3 + T \\
    \sigma_{ad} &= 0 \\
    w_4 &= c_4 \\
    \sigma_4 q_4 &= 0 \\
    w_{ce} &= c_4 + c_5 + T \\
    \sigma_{ce} &= 0 \\
    w_5 &= c_5 \\
    \sigma_5 q_5 &= 0
\end{align*}
\]

commodity set: \( \mathcal{K} = \{ a \to d, c \to e \} \)
2. Motorway Network with alternative, toll-free trunk roads

Model examples: underlying Graph

TOLL EVASION LAYER

TRANSITION EDGES ($\sigma = 0$, transition cost $\tau$)

MOTORWAY & SHORTCUTS ($\sigma = 0$)
Spot-checking games

Given:
DiGraph $G = (V, E)$ with edge weights $(w_e, \sigma_e, \beta_e)_{e \in E}$,
and a set of commodities $C = \{(s_k, t_k, x_k) : k \in K\}$.

We use the standard approximation

$$1 - \prod_{e \in r} (1 - q_e \pi_e) \simeq \sum_{e \in r} q_e \pi_e$$

Expected loss for network user choosing path $r$

$$\Pi_k(r) = -\left(\sum_{e \in r} w_e + q_e \sigma_e \right)$$
Players and Payoffs in a SC game

- The network user chooses a multicommodity flow $p \in \mathcal{F}$
- The inspector chooses control intensities $q \in Q$
- Payoff for the network user:

$$\Pi_U = -\sum_e p_e (w_e + \sigma_e q_e)$$

- Payoff for the inspector:

$$\Pi_I = \sum_e p_e (\beta_e + \alpha \sigma_e q_e)$$

Typically,
- $\beta_e \leq w_e$.
- $\alpha = 1$ (MAXPROFIT) or $\alpha = 0$ (MAXTOLL)
Set of Inspector’s strategies $\mathcal{Q}$

1. Ideal distribution of $\gamma$ controllers

$$\mathcal{Q} = \{ q \in [0, 1]^E : \sum_e q_e = \gamma \}$$

2. Feasible Duty rosters

Define $\mathcal{Q}$ as the set of unit flows in a cyclic duty graph:
Outline

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3. The price of Spite

4. Numerical Results
Game Theory Background

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Best response strategies

$p^* \in BR(q) \iff \forall k \in K, \forall p'_k \in P_k, \quad \Pi_k(p^*_k, q) \geq \Pi_k(p'_k, q)$; $q^* \in BR(p) \iff \forall q' \in Q, \quad \Pi_I(p, q^*) \geq \Pi_I(p, q')$;
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### Best response strategies

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### Nash Equilibrium

$(p^*, q^*)$ is a *Nash equilibrium* iff $p^* \in BR(q^*)$ and $q^* \in BR(p^*)$. 

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**G. Sagnol**

Spot Checking Games, SAGT 2014
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Nash Equilibrium

$(p^*, q^*)$ is a Nash equilibrium iff $p^* \in BR(q^*)$ and $q^* \in BR(p^*)$.

Strong Stackelberg equilibrium

$(p^*, q^*)$ is a SSE iff $(p^*, q^*) \in \arg \max_{p \in BR(q)} \Pi_I(p, q)$. 

G. Sagnol

Spot Checking Games, SAGT 2014 10/24
Nash Equilibria

Theorem
Although non zero-sum, a Nash equilibrium of the spot-checking game can be found by Linear Programming:

\[
\begin{align*}
\max_{q, \lambda, y} \quad & \sum_{k} x_k \lambda_k \\
\text{s. t.} \quad & y^s_v - y^s_u \leq w(u,v) + \sigma(u,v) q(u,v), \quad \forall s \in V, \forall (u, v) \in E; \quad (1b) \\
& y^s_s = 0, \quad \forall s \in V; \quad (1c) \\
& \lambda_k \leq y^\text{src}(k) \quad \forall k \in K \quad (1d) \\
& q \in Q \quad (1e)
\end{align*}
\]
Theorem
Although non zero-sum, a Nash equilibrium of the spot-checking game can be found by Linear Programming:

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\begin{align*}
\max_{q, \lambda, y} & \quad \sum_k x_k \lambda_k \left( + \sum_e p_e (\beta_e - \alpha w_e) \right) \\
\text{s. t.} & \quad y_v^s - y_u^s \leq w_{(u,v)} + \sigma_{(u,v)} q_{(u,v)}, \quad \forall s \in V, \forall (u, v) \in E; \\
& \quad y_s^s = 0, \quad \forall s \in V; \\
& \quad \lambda_k \leq y_{\text{src}(k)}^{\text{dst}(k)}, \quad \forall k \in K \\
& \quad q \in Q
\end{align*}
\]
**Matrix form of SC games**

Observe that SC games can be expressed as bimatrix games (modulo the fact that $p$ and $q$ belong to polyhedra rather than standard probability simplexes):

$$\Pi_U = - \sum_e p_e (w_e + \sigma_e q_e) = -p^T \left( \frac{1}{\gamma} w 1^T + \text{diag}(\sigma) \right) q$$

$$\Pi_I = \sum_e p_e (\beta_e + \sigma_e q_e) = p^T \left( \frac{1}{\gamma} \beta 1^T + \alpha \text{diag}(\sigma) \right) q$$

For $\alpha = 1$, the game has a nice “zero-sum + costs” structure:

$$\Pi_U + \Pi_I = \sum_e p_e (\beta_e - w_e)$$
Complexity of polymatrix games

- It is known that NE of \textit{pairwise zero-sum} games can be computed by linear programming (extension of the minimax theorem, [Cai & Daskalakis, 2011])
- It is known that it is NP hard to compute the SSE of a polymatrix game [Conitzer & Sandholm, 2011]
- But what about SSE for \textit{pairwise zero-sum} polymatrix games?
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**Theorem**

It is NP hard to compute a SSE in a SC game, even when $\alpha = 1$ and every user has only 2 possible routes.
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- But what about SSE for *pairwise zero-sum* polymatrix games?

**Theorem**

It is NP hard to compute a SSE in a SC game, even when \( \alpha = 1 \) and every user has only 2 possible routes.

**Corollary**

It is NP hard to compute a SSE in a polymatrix game, even when the game is pairwise zero-sum.
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Efficiency of a strategy: Inspector payoff in the Stackelberg model:

\[ H(q) := \max_{p \in BR(q)} \sum_{e \in E} p_e(\alpha \sigma_e q_e + \beta_e). \]

Define \( N \subseteq Q \): set of NE strategies of the inspector:

\( q \) is in \( N \) if it maximizes \( \sum_k x_k \text{ spl}_k(w_e + \sigma_e q_e) \).

**Definition**

Price of spite:

\[ \text{PoS} := \frac{\max_{q \in Q} H(q)}{\min_{q \in N} H(q)}. \]
An *a posteriori* bound

(For simplicity, we assume $\alpha = 1$)

For a feasible multicommodity flow $p \in \mathcal{F}$, define

$$d(p) = \sum_{e} p_e (w_e - \beta_e)$$

Define $d_{\text{min}} := \min\{d(p) : p \in \mathcal{F}\}$.

**Proposition**

Let $\mathcal{G}$ be a SC game such that $w_e \geq \beta_e$ for all $e \in E$. Let:

- $(p^*, q^*)$ be a NE of $\mathcal{G}$;
- $\hat{p}$ be a tie-breaking best response to $q^*$.

Then, for all $q \in \mathcal{Q}$ we have:

$$H(q) \leq H(q^*) + d(\hat{p}) - d_{\text{min}}.$$
An instance with unbounded PoS

Edge labelled with $(w_e, \sigma_e, \beta_e)$.

$Q = \{0 \leq q \leq 1 : \sum_e q_e = 2\}$. 
Assumptions for a distance-based toll

So we need additional assumptions to bound the PoS. Denote by $l_e$ the length of edge $e$, and set $\ell_k := \text{spl}_k(l_e)$.

1. **Partition of the edge set in three classes:**

\[
\begin{align*}
we &= (b + f)l_e, \quad \beta_e = fl_e, \quad \sigma_e = 0 & \text{if } e \in E_P \quad \text{(pay edge)} \\
we &\geq bl_e, \quad \beta_e = 0, \quad \sigma_e > 0 & \text{if } e \in E_E \quad \text{(evasion edge)} \\
w_e &\geq 0, \quad \beta_e = \sigma_e = l_e = 0 & \text{if } e \in E_D \quad \text{(dummy edge)}
\end{align*}
\]

2. For each commodity, there is a $(s_k, t_k)$—path $R^k_{\text{pay}}$ with no evasion edge, s.t. $\sum_{e \in R^k_{\text{pay}}} w_e = (b + f)\ell_k$.

3. There exists a uniform control strategy $q^U \in Q$ on the evasion edges, and $q^U_e$ is proportional to $\sigma_e^{-1}l_e$ on the evasion edges, that is, $\exists u > 0 : \forall e \in E_E, q^U_e = u \frac{l_e}{\sigma_e}$. 
A parametrized bound

**Theorem**
Consider an SC game $\mathcal{G}$ satisfying assumptions (A1)-(A3). Then, the price of spite of $\mathcal{G}$ is bounded from above by $\max \left(1, \frac{f}{\alpha u}\right)$. 

A parametrized bound

Theorem
Consider an SC game $G$ satisfying assumptions (A1)-(A3). Then, the price of spite of $G$ is bounded from above by $\max \left(1, \frac{f}{\alpha u} \right)$.

Proof:
- Let $\mathcal{L} := \sum_k x_k \ell_k$ be the minimum of total distance covered.
- For all $q \in Q$,
  $$H(q) \leq f \mathcal{L}.$$
- Let $(p^*, q^*)$ be a NE:
  $$H(q^*) \geq \Pi_I(p^*, q^*) = \max_{q \in Q} \sum_{e \in E} p_e^* (\alpha \sigma_e q_e + \beta_e) \geq \min_{p \in F} \max_{q \in Q} \sum_{e \in E} p_e (\alpha \sigma_e q_e + \beta_e)$$
  $$= \max_{q \in Q} \min_{p \in F} \sum_{e \in E} p_e (\alpha \sigma_e q_e + \beta_e)$$
  $$\geq \min_{p \in F} \alpha u \sum_{e \in E_E} p_e \ell_e + f \sum_{e \in E_P} p_e \ell_e$$
  $$\geq \min(\alpha u, f) \mathcal{L}.$$
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Experimental settings

- The Motorway network is divided into control regions. We solve the problem independently on each region.
- Real traffic data (averaged over time for the static instances).
- All instances solved with CPLEX.
- We compare the results to the naive solution, where controls are proportional to the traffic volume on each edge.
- For the static instances, the results are presented for the model with one toll-edge for each commodity, and

\[ Q = \{ \mathbf{q} \in [0,1]^E : \sum_e q_e = \gamma \} . \]
Optimal control strategies

Solution of the Nash LP, $\gamma = 50$ inspector teams:

Germany
static instance:
319 nodes,
2948 edges,
5013 commodities.

“Efficiency” of this strategy (w.r.t. SSE, $\alpha = 1$) $\geq 99.3\%$.
Upper bound on the PoS: $\frac{f}{u} = 1.43$
## Bounds for several instances

Nash LP solved for the Two-layer, time extended model.
Conclusion and perspectives

Conclusion

- A flexible framework to model toll enforcement problems in transportation networks
- LP and MIP formulation to compute Nash and Stackelberg strategies
- For a distance-based toll, the Nash LP remains tractable for very large instances and approximates the Stackelberg strategy that optimizes the total revenue.

Future work

- Tightness of bound, better bound on PoS ?
- Is there a FPTAS for SSE when $\alpha = 1$, and more generally in pairwise zero-sum polymatrix game ?
- Congestion