Optimizing Toll Enforcement in Transportation Networks: a Game-Theoretic Approach

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Truck toll in Germany

A Huge and original system

- Started in 2005
- Tax based on distance and truck category (avg price: 0.17 Euro/km)
- Total revenue around 3 Billion Euros / year
- No toll barriers
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Toll Enforcement

- Automatic controls: 300 toll checker gantries (Kontrollbrücke)
- Mobile control units: 300 vehicles - 540 officers from BAG (Federal Office of Goods Transport)
Optimizing Toll Enforcement

Cooperation project between ZIB and BAG

Development of a tool to generate
- feasible duty rosters
- optimal tour planning

Specifications

Mobile inspection teams must carry out a network-wide control whose intensity is proportional to given spatial and time dependent traffic distributions.
Outline

1. Problem formulation
   - Players and Payoffs
   - Model examples

2. Computation of Equilibria
   - Nash Equilibria for MAXPROFIT
   - A MIP formulation for Stackelberg Equilibria

3. Numerical Results
Outline

1 Problem formulation
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3 Numerical Results
A simple, general model

Notation

$G(V, E)$ : Graph of the network
$\mathcal{K}$ : set of commodities (OD pairs) on the network
$x_k$ : nb of users (trucks) of commodity $k \in \mathcal{K}$
$w_e$ : cost for taking $e \in E$ (travel charge + toll fare)
A simple, general model

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\[ w_e : \text{cost for taking } e \in E \text{ (travel charge + toll fare)} \]
\[ q_e : \text{Probability that an inspector is on section } e \in E \]
\[ \sigma_e : \text{Probability to be controlled on } e, \text{ conditionally to the presence of an inspector} \]
A network spot-checking game

Given:
DiGraph $G(V, E)$ with costs and control risks $(w_e, \sigma_e)_{e \in E}$, commodities $\mathcal{K} = \{k_1, \ldots, k_m\}$ with traffic $(x_k)_{k \in \mathcal{K}}$, amount $P$ of the penalty if a toll evader is caught.
A network spot-checking game

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Assuming:
- stationary regime: users of commodity $k$ have learnt the probabilities $\pi_r$ to be controlled on each path $r \in \mathcal{R}_k$.
- selfish, rational drivers: Choose their path $r \in \mathcal{R}_k$ so as to minimize their expected loss $\sum_{e \in r} w_e + \pi_r P$. 

Goal:
Distribute the controls $q$ over the edges of the network, to maximize the expected revenue (fares + penalties) OR minimize the number of evaders (or evaded kilometers).
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- **OR** minimize the number of evaders (or evaded kilometers)
Players of the spot-checking game

- “Player $k$” represents all users of commodity $k$.
  - Commits to a mixed strategy $\hat{p}^k$, i.e. chooses path $r$ with probability $\hat{p}^k_r$. 

"The Inspector" represents the set of controllers.

- Commits to a strategy $q \in Q \subseteq [0,1]^E$. 

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  - This defines a unit flow $p^k$ through commodity $k$:

$$p^k_e := \sum_{r \ni e} \hat{p}^k_r.$$
Players of the spot-checking game

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  - Commits to a mixed strategy \( \hat{p}^k \), i.e. chooses path \( r \) with probability \( \hat{p}_r^k \).
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    \[
p_e^k := \sum_{r \ni e} \hat{p}_r^k.
    \]

- **“The Inspector”** represents the set of controllers.
  - Commits to a strategy \( q \in Q \subset [0, 1]^E \).
Game Theory Background

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Best response strategies

$p^* \in BR(q)$ $\iff \forall k \in K, \forall p'_k \in P_k, \Pi_k(p^*_k, q) \geq \Pi_k(p'_k, q)$

$q^* \in BR(p)$ $\iff \forall q' \in Q, \Pi_C(p, q^*) \geq \Pi_C(p, q')$

Nash Equilibrium $(p^*, q^*)$ is a Nash equilibrium iff $p^* \in BR(q^*)$ and $q^* \in BR(p^*)$.

Stackelberg equilibrium $(p^*, q^*)$ is a Stackelberg equilibrium iff $(p^*, q^*) \in \text{argmax}_{p \in BR(q^*)} \Pi_C(p, q^*)$.
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**Best response strategies**

\[
p^* \in BR(q) \iff \forall k \in K, \forall p'_k \in P_k, \quad \prod_k(p^*_k, q) \geq \prod_k(p'_k, q);
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q^* \in BR(p) \iff \forall q' \in Q, \quad \prod_C(p, q^*) \geq \prod_C(p, q');
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Best response strategies

$p^* \in BR(q) \iff \forall k \in K, \forall p'_k \in P_k, \quad \Pi_k(p^*_k, q) \geq \Pi_k(p'_k, q)$;

$q^* \in BR(p) \iff \forall q' \in Q, \quad \Pi_C(p, q^*) \geq \Pi_C(p, q')$;

Nash Equilibrium

$(p^*, q^*)$ is a Nash equilibrium iff $p^* \in BR(q^*)$ and $q^* \in BR(p^*)$. 
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### Best response strategies

$p^* \in BR(q) \iff \forall k \in K, \forall p'_k \in P_k$, $\Pi_k(p^*_k, q) \geq \Pi_k(p'_k, q)$

$q^* \in BR(p) \iff \forall q' \in Q$, $\Pi_C(p, q^*) \geq \Pi_C(p, q')$

### Nash Equilibrium

$(p^*, q^*)$ is a *Nash equilibrium* iff $p^* \in BR(q^*)$ and $q^* \in BR(p^*)$.

### Stackelberg equilibrium

$(p^*, q^*)$ is a *Stackelberg equilibrium* iff $(p^*, q^*) \in \arg\max_{p \in BR(q)} \Pi_C(p, q)$. 
Users’ best response strategies

Probability to be controlled on path $r$

$\pi_r = 1 - \prod_{e \in r} (1 - q_e \sigma_e)$

\[ \approx \sum_{e \in r} q_e \sigma_e \]
Users’ best response strategies

Probability to be controlled on path $r$

$\pi_r = 1 - \prod_{e \in r} (1 - q_e \sigma_e)$

$\approx \sum_{e \in r} q_e \sigma_e$

Expected loss of Player $k$ when choosing path $r$

$\Pi_k(r) = - \left( \sum_{e \in r} w_e + q_e \sigma_e P \right)$
Users' best response strategies

Probability to be controlled on path $r$

Expected loss of Player $k$ when choosing path $r$

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\pi_r = 1 - \prod_{e \in r} (1 - q_e \sigma_e)
\approx \sum_{e \in r} q_e \sigma_e
$$

$$\Pi_k(r) = - \left( \sum_{e \in r} w_e + q_e \sigma_e P \right)$$

Best strategy for Player $k$ given $q \in Q$: Choose a shortest path in $G(V, E)$ with augmented weights $w'_e := w_e + q_e \sigma_e P$. 
Inspector’s Payoff

- Reward $\beta_e$ each time a user takes edge $e$
- Fraction $\alpha$ of all collected Penalties

$$\Pi_C(p, q) = \sum_{k \in \mathcal{K}} x_k \left( \sum_{e \in E} p_e^k (\beta_e + \alpha \sigma_e q_e P) \right)$$

Particular cases

- **MAXPROFIT**: $\alpha = 1$, $\beta_e = \text{toll fare for edge } e$.
- **MAXTOLL**: $\alpha = 0$, $\beta_e = \text{toll fare for edge } e$.
- **MINEVADERS**: $\alpha = 0$, $\beta_e = 1$ when $e$ is an edge representing the option to pay the toll for the whole trip through a commodity.
Inspector’ s Payoff

- Reward $\beta_e$ each time a user takes edge $e$
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**Particular cases**

- **MAXPROFIT:** $\alpha = 1$, $\beta_e =$ toll fare for edge $e$.
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1. Transit system with fixed access fare $T$. 

$w_1 = c_1$  
$\sigma_1 q_1$  

$w_2 = c_2$  
$\sigma_2 q_2$  

$w_3 = c_3$  
$\sigma_3 q_3$  

$w_4 = c_4$  
$\sigma_4 q_4$  

$w_5 = c_5$  
$\sigma_5 q_5$
1. Transit system with fixed access fare $T$.

Model examples: underlying Graph

commodity set: $\mathcal{K} = \{a \rightarrow d, c \rightarrow e\}$
Model examples: underlying Graph

2. Motorway Network with alternative, toll-free trunk roads
Set of Inspector’s strategies $Q$

1. Ideal distribution of $\gamma$ controllers

$$Q = \{ q \in [0, 1]^E : \sum_e q_e = \gamma \}$$

2. Feasible Duty rosters

Define $Q$ as the set of unit flows in a cyclic duty graph:
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3 Numerical Results
Nash Equilibria for MAXPROFIT

Theorem
Although non zero-sum, a Nash equilibrium of the MAXPROFIT spot-checking game can be found by Linear Programming:

\[
\begin{align*}
\text{max}_{q, \lambda, y} & \quad \sum_k x_k \lambda_k \\
\text{s.t.} & \quad y^s_v - y^s_u \leq w_{(u,v)} + \sigma_{(u,v)} q_{(u,v)} P, \quad \forall s \in V, \, \forall (u, v) \in E; \\
& \quad y^s_s = 0, \quad \forall s \in V; \\
& \quad \lambda_k \leq y^{\text{src}(k)}_{\text{dst}(k)} \quad \forall k \in K; \\
& \quad q \in Q
\end{align*}
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a MIP for Stackelberg Equilibria

Theorem

A Stackelberg equilibrium of the spot-checking game can be found by solving the following MIP

\[
\begin{align*}
\max_{q, y, \lambda, \mu, \rho} & \quad \sum_k \alpha x_k \lambda_k + \sum_{s \in V} \sum_{e \in E} \rho_s^e (\beta_e - \alpha w_e) \\
\text{s.t.} & \quad \sum_{k} \alpha x_k \lambda_k + \sum_{s \in V} \sum_{e \in E} \rho_s^e (\beta_e - \alpha w_e) \\ & \quad \sum_{v \in V} \sum_{k} x_k (s \to v) = \{ s \} \quad \forall s \in S \\
& \quad \sum_{s \in V} \sum_{e \in E} \rho_s^e (\beta_e - \alpha w_e) = \{ s \} \quad \forall s \in S \times V \\
& \quad 0 \leq \rho_s^e \leq M \mu_s^e \quad \forall (s, e) \in S \times E \\
& \quad \mu_s^e \in \{0, 1\} \quad \forall (s, e) \in S \times E.
\end{align*}
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A Stackelberg equilibrium of the spot-checking game can be found by solving the following MIP

$$\max_{q, y, \lambda, \mu, \rho} \sum_k \alpha x_k \lambda_k + \sum_{s \in V} \sum_{e \in E} \rho^s_e (\beta_e - \alpha w_e)$$

$$0 \leq w(u, v) + \sigma(u, v) q(u, v) P - (y_v^s - y_u^s) \quad \forall s \in V, \forall (u, v) \in E;$$

$$y_s^s = 0, \quad \forall s \in S;$$

$$\lambda_k \leq y_{\text{src}(k)}^s, \quad \forall k \in K;$$

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a MIP for Stackelberg Equilibria

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& \quad 0 \leq w(u, v) + \sigma(u, v) q(u, v) P - (y^s_v - y^s_u) \quad \forall s \in V, \forall (u, v) \in E; \\
& \quad y^s_s = 0, \quad \forall s \in S; \\
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\begin{align*}
\max_{q,y,\lambda,\mu,\rho} & \quad \sum_k \alpha x_k \lambda_k + \sum_{s \in V} \sum_{e \in E} \rho_s^e (\beta_e - \alpha w_e) \\
0 & \leq w(u,v) + \sigma(u,v) q(u,v) P - (y_v^s - y_u^s) \leq M(1 - \mu_{s,(u,v)}^s), \quad \forall s \in V, \forall (u,v) \in E; \\
y_s^s = 0, \quad \forall s \in S; \\
\lambda_k & \leq y_{\text{src}(k)}, \quad \forall k \in K; \\
q & \in Q \\
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\mu_e^s \in \{0, 1\}, \quad \forall (s,e) \in S \times E.
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Theorem

A Stackelberg equilibrium of the spot-checking game can be found by solving the following MIP

\[
\max_{q, y, \lambda, \mu, \rho} \quad \sum_k \alpha x_k \lambda_k + \sum_{s \in V} \sum_{e \in E} \rho^s_e (\beta_e - \alpha w_e)
\]

\[
0 \leq w(u, v) + \sigma(u, v) q(u, v) P - (y^s_v - y^s_u) \leq M(1 - \mu^s_{(u,v)}), \quad \forall s \in V, \forall (u, v) \in E;
\]

\[
y^s_s = 0,
\]

\[
\lambda_k \leq y^\text{src}(k), \quad \forall k \in K;
\]

\[
q \in Q
\]

\[
\sum_{\{u:(v,u)\in E\}} \rho^s_{(v,u)} - \sum_{\{u:(u,v)\in E\}} \rho^s_{(u,v)} = \begin{cases} 
\sum_v x_{(s \rightarrow v)} & \text{if } s = v; \\
-x_{(s \rightarrow v)} & \text{if } (s \rightarrow v) \in K; \\
0 & \text{otherwise,}
\end{cases} \quad \forall (s, v) \in S \times V;
\]

\[
0 \leq \rho^s_e \leq M \mu^s_e, \quad \forall (s, e) \in S \times E;
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3. Numerical Results
Experimental settings

- The Motorway network is divided into control regions. We solve the problem independently on each region.
- Real traffic data (averaged over time).
- All instances solved with CPLEX.
- We compare the results to the naive solution, where controls are proportional to the traffic volume on each edge.
- All the results are presented for the model with one toll-edge for each commodity, and \[ \mathcal{Q} = \{ \mathbf{q} \in [0, 1]^E : \sum_e q_e = \gamma \} \].
Optimal control strategies

Optimum of the Nash LP, MAXPROFIT setting, $\gamma = 3$ inspectors:

Region of North Rhine-Westfalia:
111 nodes, 264 directed edges, 4905 commodities.
Optimal control strategies

Near optimum of the Stackelberg MIP (gap = 1.5%), MAXPROFIT setting, $\gamma = 50$ inspectors:

Whole Germany:
319 nodes, 2948 edges, 5013 commodities
Comparison of different equilibria

Region of Berlin-Brandenburg (|V| = 45, |E| = 130, |K| = 596)

Revenue vs. Number of controllers
Comparison of different equilibria

Region of Berlin-Brandenburg (|V| = 45, |E| = 130, |K| = 596)

Revenue vs. Number of controllers

G. SAGNOL

INOC 2013
Comparison of different equilibria

Region of Berlin-Brandenburg ($|V| = 45, |E| = 130, |K| = 596$)

Toll Payment rate vs. Number of controllers

G. Sagnol

INOC 2013
Comparison of different equilibria

Region of Berlin-Brandenburg ($|V| = 45$, $|E| = 130$, $|K| = 596$)

Effect of $\alpha$ on the fraction of toll in Revenue
Conclusion and perspectives

Conclusion

- A flexible framework to model toll enforcement problems in transportation networks
- LP and MIP formulation to compute Nash and Stackelberg strategies
- The Nash LP remains tractable for very large instances and seems to approximate the Stackelberg strategy that optimizes the total revenue.

Future work

- Integrate this approach in our duty roster optimizer
- Investigate theoretical bounds on the gap between Nash and Stackelberg Strategies (for MAXPROFIT)