Sobolev estimates for spherical means and the Funk–Radon transform

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Content

Funk–Radon transform

[Funk 1911]

- Sphere $\mathbb{S}^{d-1} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$
- Funk–Radon transform

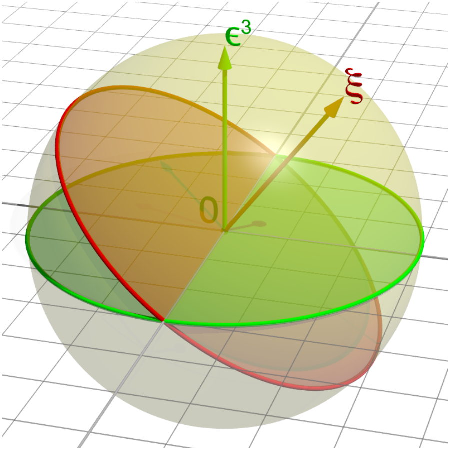
$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 0} f(\boldsymbol{\eta})\, \mathrm{d}\lambda(\boldsymbol{\eta})$$

(integrals of f along all great circles)

Goal

Reconstruct the function f from integrals $\mathcal{F}f$

- Possible for even functions $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$



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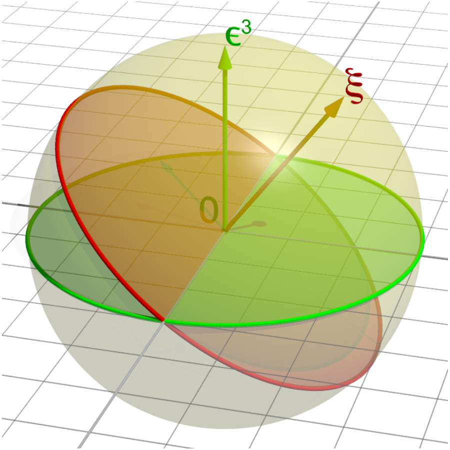

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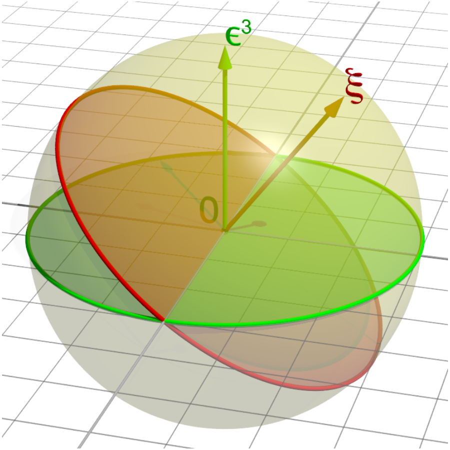

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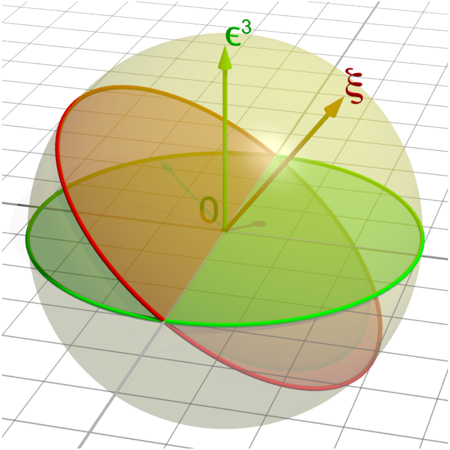

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- Y_n^k ... spherical harmonic of degree $n \in \mathbb{N}_0$
- Any $f \in L^2(\mathbb{S}^{d-1})$ can be represented as Fourier series

$$f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \langle f, Y_n^k \rangle \, Y_n^k$$

- **Sobolev space** $H^s(\mathbb{S}^{d-1})$ of order $s \in \mathbb{R}$ is the completion of the space of smooth functions $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ with the norm

$$\|f\|^2_{H^s(\mathbb{S}^{d-1})} = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\langle f, Y_n^k \rangle\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s}$$
$$= \left\| \left(-\Delta^\bullet + \frac{(d-2)^2}{4}\right)^{s/2} f \right\|^2_{L^2(\mathbb{S}^{d-1})}$$

- $\Delta^\bullet$ ... Laplace–Beltrami operator on $\mathbb{S}^{d-1}$

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Sobolev estimate of the Funk–Radon transform

Lemma (eigenvalue decomposition) [Minkowski 1904] for $d = 3$

The Funk–Radon transform $\mathcal{F}$ satisfies

$$\mathcal{F}Y_n^k(\boldsymbol{\xi}) = P_{n,d}(0) Y_n^k(\boldsymbol{\xi}), \qquad P_{n,d}(0) = \begin{cases} \dfrac{(-1)^{n/2}\,(n-1)!!\,(d-3)!!}{(n+d-3)!!}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

$P_{n,d}$ – Legendre (ultraspherical) polynomial of degree n in dimension d

Theorem [Strichartz 1981]

The Funk–Radon transform $\mathcal{F}$ extends to a bounded, bijective and open operator

$$\mathcal{F}\colon H^s_{\text{even}}(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}_{\text{even}}(\mathbb{S}^{d-1}).$$

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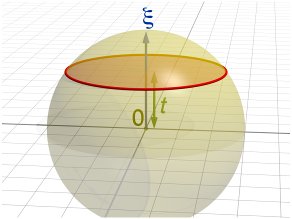
Circular means on the sphere



- $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$
- **Mean operator** integrates f along all hyperplane sections:

$$\mathcal{M}f(\boldsymbol{\xi}, t) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = t} f(\boldsymbol{\eta}) \, \mathrm{d}\lambda(\boldsymbol{\eta}), \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1}, t \in [-1, 1]$$

- The inversion of $\mathcal{M}$ is overdetermined
  e.g. $\mathcal{M}f(\xi, 1) = f(\xi)$
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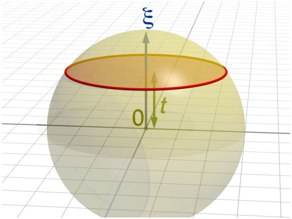

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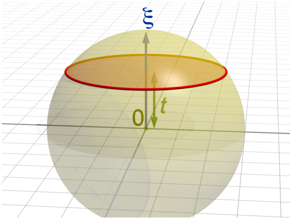

Singular value decomposition [Berens, Butzer & Pawelke 1961]

- Y_n^k spherical harmonic of degree n
- $P_{n,d}$ Legendre (ultraspherical) polynomial of degree n in dimension d, orthogonal polynomial on $[-1, 1]$ w.r.t. the weight $(1 - t^2)^{\frac{d-3}{2}}$

Then

$$\mathcal{M}Y_n^k(\boldsymbol{\xi}, t) = Y_n^k(\boldsymbol{\xi}) P_{n,d}(t).$$

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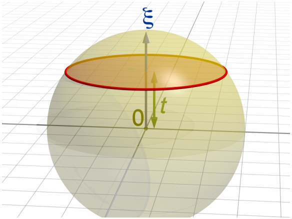

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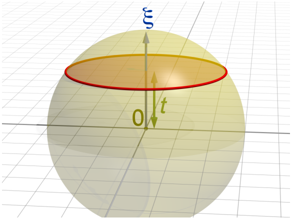

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Sobolev space on $\mathbb{S}^{d-1} \times [-1,1]$

- Reminder: Sobolev norm on the sphere $\mathbb{S}^{d-1}$

$$\|f\|^2_{H^s(\mathbb{S}^{d-1})} = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\langle f, Y_n^k \rangle\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s}$$

- Sobolev norm in $H^{s,r}(\mathbb{S}^{d-1} \times [-1,1])$ for $s, r \in \mathbb{R}$

$$\|g\|^2_{H^{s,r}(\mathbb{S}^{d-1} \times [-1,1]; w_d)} = \sum_{n,l=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\langle g, Y_n^k \widetilde{P}_{l,d} \rangle\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s} \left(l + \tfrac{d-2}{2}\right)^{2r}$$

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Sobolev estimate of $\mathcal{M}$

Theorem

Let $s \in \mathbb{R}$. The mean operator $\mathcal{M}$ on the sphere $\mathbb{S}^{d-1}$ extends to a bounded linear operator

$$\mathcal{M}\colon H^s(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times [-1,1]; w_d).$$

Injectivity sets of $\mathcal{M}$

Theorem [Hielscher, Q.]

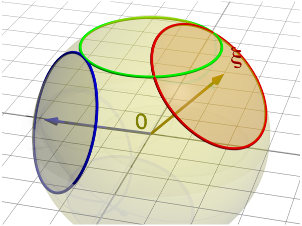
Let $D \subset \mathbb{S}^{d-1} \times [-1,1]$, $g_0 \colon D \to \mathbb{C}$, and let $s > \frac{d-1}{2}$. The following are equivalent:

1. The problem
$$\mathcal{M}\big|_D f = g_0$$
has a unique solution $f \in H^s(\mathbb{S}^{d-1})$.
2. The Euler–Poisson–Darboux differential equation
$$\Delta^{\bullet}_{\boldsymbol{\xi}} g(\boldsymbol{\xi}, t) = \left((1-t^2) \frac{\partial^2}{\partial t^2} - (d-1)\, t \frac{\partial}{\partial t} \right) g(\boldsymbol{\xi}, t).$$
with boundary condition $g\big|_D = g_0$ has a unique solution
$$g \in H^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times [-1,1]; w_d).$$

Circles with fixed radius

For fixed $z \in [-1, 1]$, compute

$$\mathcal{T}_z f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = z} f(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta}$$



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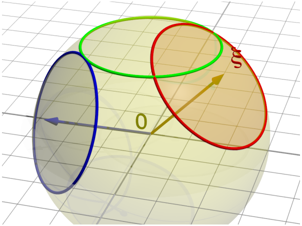

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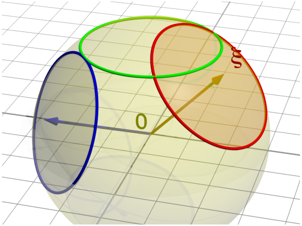

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“Freak theorem” [Schneider 1969]

The set of values z for which $\mathcal{T}_z$ is **not** injective is countable and dense in $[-1, 1]$.

This is because $\mathcal{T}_z$ is injective if and only if $P_{n,d}(z) = 0 \; \forall n \in \mathbb{N}_0$.

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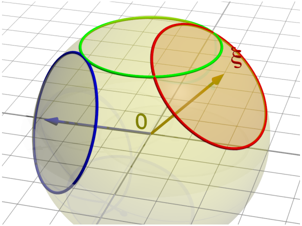

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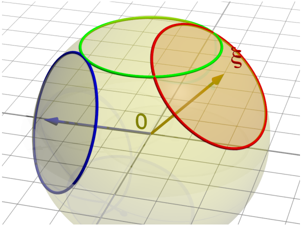

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Sobolev estimate

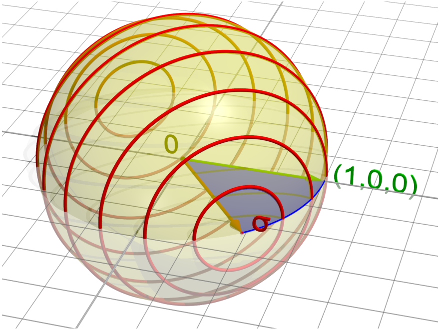
Let $z \in (-1,1)$ and $s \in \mathbb{R}$. The spherical section transform $\mathcal{T}_z$ is a bounded operator

$$\mathcal{T}_z \colon H^s(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}(\mathbb{S}^{d-1}).$$

However, the range is only a subset of $H^{s+\frac{d-2}{2}}(\mathbb{S}^{d-1})$. [Rubin 2000]

Vertical slices

$$\mathcal{V}f(\boldsymbol{\sigma},t)=\mathcal{M}f\left(\binom{\boldsymbol{\sigma}}{0},t\right)=\int_{\langle\binom{\boldsymbol{\sigma}}{0},\boldsymbol{\eta}\rangle=t} f(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta}),\qquad \boldsymbol{\sigma}\in\mathbb{S}^{d-2}$$



- **Circles perpendicular to the equator**
- Injective for symmetric functions $f(\xi_1,\xi_2,\xi_3)=f(\xi_1,\xi_2,-\xi_3)$
- Orthogonal projection to equatorial plane
  ↗ Radon transform in $\mathbb{R}^2$
  [Gindikin, Reeds & Shepp 1994]
- Application in photoacoustic tomography
  [Zangerl & Scherzer 2010]
- Singular value decomposition
  [Hielscher & Q. 2016] [Rubin 2018]

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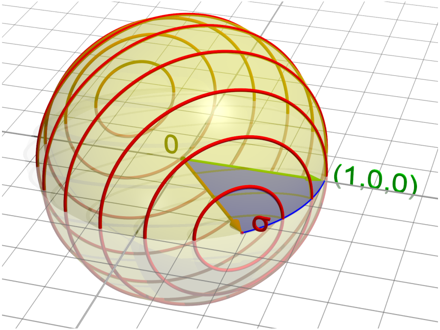

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- Orthogonal projection to equatorial plane ↗ Radon transform in $\mathbb{R}^2$ [Gindikin, Reeds & Shepp 1994]
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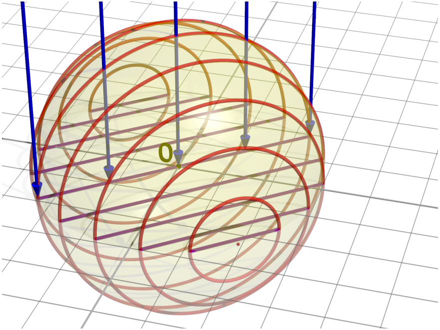

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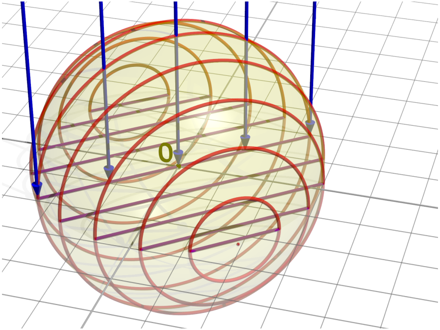

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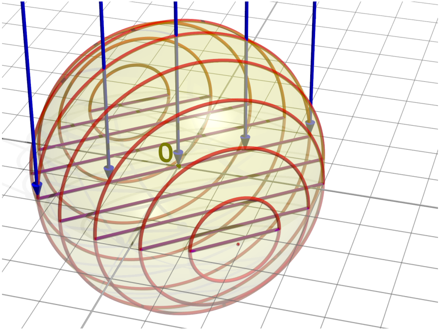

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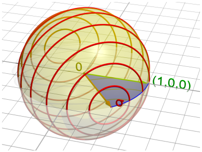

Sobolev estimate [Q.]

The vertical slice transform

$$\mathcal{V}\colon H^s(\mathbb{S}^{d-1}) \to H_{\mathrm{even}}^{0,s+\frac{d-2}{2}-\frac{1}{4}}(\mathbb{S}^{d-2} \times [-1,1]; w_d)$$

is a bounded operator.

$\mathcal{V}$ is not onto. We have for the range

$$H_{\mathrm{even}}^{0,s+\frac{d-2}{2}}(\mathbb{S}^{d-2} \times [-1,1]) \subset \mathrm{range}(\mathcal{V}) \subset H_{\mathrm{even}}^{0,s+\frac{d-2}{2}-\frac{1}{4}}(\mathbb{S}^{d-2} \times [-1,1]).$$

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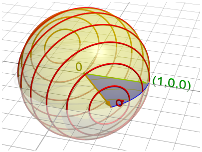

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Planes through a fixed point

[Salman 2016]

Consider an arbitrary point inside the sphere:

$$(0,\dots,0,z), \qquad 0 \le z < 1$$

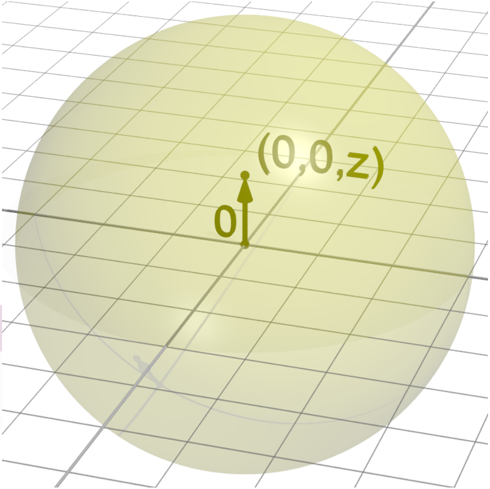
Plane section through $(0,\dots,0,z)$ is

$$\{\boldsymbol{\eta} \in \mathbb{S}^{d-1} : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = z\xi_d\}.$$

Definition

$$\mathcal{U}_z f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = z\xi_d} f(\boldsymbol{\eta}) \,\mathrm{d}\lambda(\boldsymbol{\eta})$$

$z = 0$: Funk–Radon transform



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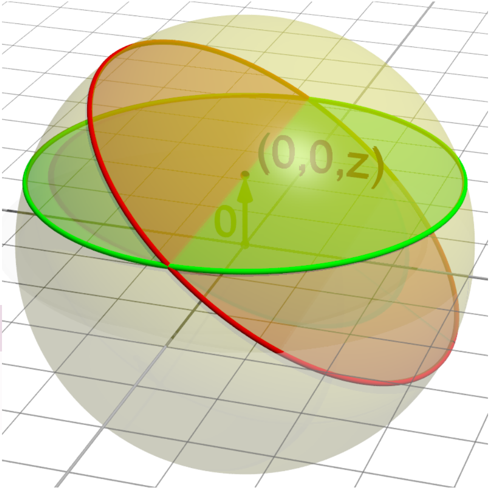

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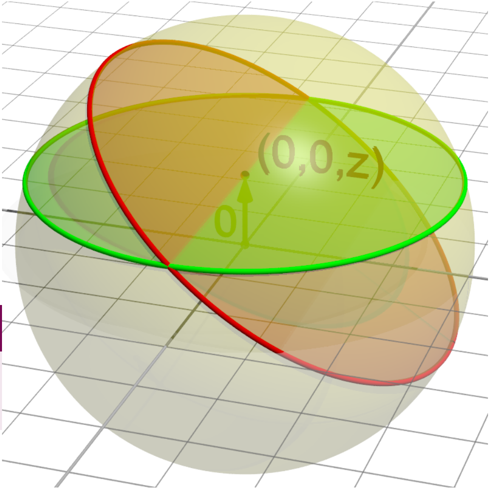

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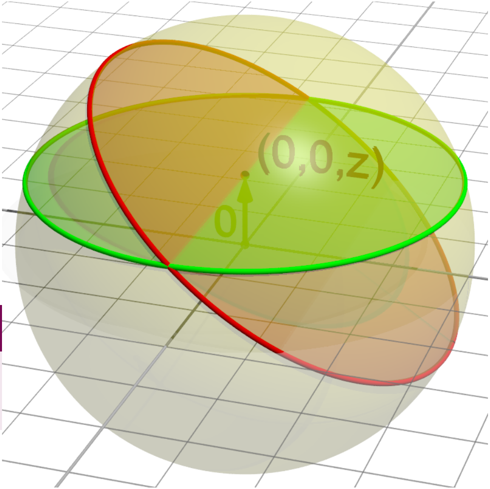


Planes through a fixed point

Nullspace of $\mathcal{U}_z$ [Q. 2018]

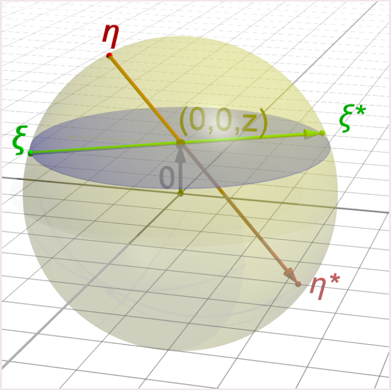
For $\xi \in \mathbb{S}^{d-1}$, set $\xi^* \in \mathbb{S}^{d-1}$ as the point reflection of the sphere about the point $(0, \dots, 0, z)$.

Let $f \in L^2(\mathbb{S}^{d-1})$. Then

$$\mathcal{U}_z f = 0$$

if and only if for almost every $\xi \in \mathbb{S}^{d-1}$

$$f(\xi) = -\frac{1 - z^2}{1 + z^2 - 2z\eta_d} f(\xi^*).$$



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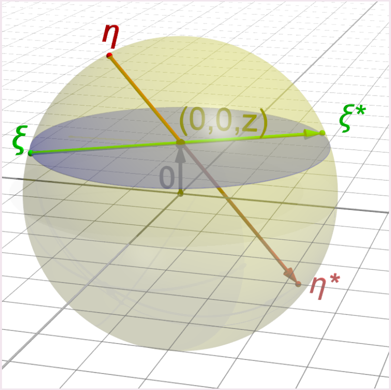

Planes through a fixed point

Sobolev estimate of $\mathcal{U}_z$ [Q. 2018]

Let $z \in (0,1)$ and $s \in \mathbb{R}$.
Set $H^s_{\mathrm{even},z}(\mathbb{S}^{d-1})$ as the subspace of all functions $f \in H^s(\mathbb{S}^{d-1})$ that satisfy

$$f(\boldsymbol{\omega}) = \left(\frac{1-z^2}{1-2z\omega_d+z^2}\right)^{d-2} f(\boldsymbol{\omega}^*), \qquad \boldsymbol{\omega} \in \mathbb{S}^{d-1},$$

almost everywhere.
Then the spherical transform

$$\mathcal{U}_z \colon H^s_{\mathrm{even},z}(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})$$

is bounded and bijective. Its inverse operator is also bounded.

Name	Definition	Injectivity	Range	SVD
mean operator	$\mathcal{M}f(\boldsymbol{\xi},t)$	✓	$\subset H^{d/2-1,0}$	✓
Funk–Radon transform	$\mathcal{M}f(\boldsymbol{\xi},0)$	$f(\boldsymbol{\xi})=f(-\boldsymbol{\xi})$	$=H^{\frac{d-2}{2}}_{\mathrm{even}}$	✓
spherical section transform	$\mathcal{M}f(\boldsymbol{\xi},z)$, $z\in[-1,1]$ fixed	✓ if $P_{n,d}(z)\neq 0$ $\forall n\in\mathbb{N}_0$	$\subset H^{\frac{d-2}{2}}$	✓
vertical slice transform	$\mathcal{M}f((\binom{\boldsymbol{\sigma}}{0}),t)$, $\boldsymbol{\sigma}\in\mathbb{S}^{d-2}$	$f(\boldsymbol{\xi}',\xi_d)=f(\boldsymbol{\xi}',-\xi_d)$	$\subset H^{0,\frac{d-2}{2}-\frac{1}{4}}$	✓
sections through fixed point	$\mathcal{M}f(\boldsymbol{\xi},z\xi_d)$, $z\in(-1,1)$ fixed	f even w.r.t. some reflection in $z\boldsymbol{\epsilon}^d$	$=\widetilde{H}_z^{\frac{d-2}{2}}$	✗
spherical slice transform	$\mathcal{M}f(\boldsymbol{\xi},\xi_d)$	✓ for $f\in L^\infty(\mathbb{S}^{d-1})$		✗

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spherical slice transform	$\mathcal{M}f(\boldsymbol{\xi},\xi_d)$	✓ for $f\in L^\infty(\mathbb{S}^{d-1})$		✗

Thank you for your attention